

$$\begin{aligned}
x(t) \in X &= \{x_1, x_2, \dots, x_n\} \\
u(t) \in U &= \{u_1, u_2, \dots, u_m\} \\
t \in T &= \{t_0, t_0 + 1, \dots, t_f\}
\end{aligned}$$

$$\begin{aligned}
\phi: X \times U &\rightarrow X & x(t+1) &= \phi(x(t), u(t)) \\
\ell: X \times U &\rightarrow \mathbb{R} & \ell(x(t), u(t)) & \\
\ell_f: X &\rightarrow \mathbb{R} & \ell_f(x(t_f)) &
\end{aligned}$$

Let $x(t_0) = x_0 \in X$ be given, and consider the optimal control problem:

$$\begin{aligned}
J(t_0, x_0; u(\cdot)) &= \sum_{\tau=t_0}^{t_f-1} \ell(x(\tau), u(\tau)) + \ell_f(x(t_f)) \\
\inf_{\substack{u(\tau) \in U \\ \tau \in T}} J(t_0, x_0; u(\cdot)) &
\end{aligned}$$

Define the cost-to-go:

$$V: T \times X \rightarrow \mathbb{R} \qquad V(t, x) = \inf_{\substack{u(\tau) \in U \\ \tau \in \{t, t+1, \dots, t_f\}}} J(t, x; u(\cdot))$$

Bellman equation:

$$\begin{aligned}
V(t_f, x) &= \ell_f(x) & \text{for all } x \in X \\
V(t, x) &= \inf_{u \in U} \{ \ell(x, u) + V(t+1, \phi(x, u)) \} & \text{for all } x \in X \text{ and all } t \in \{t_0, t_{0+1}, \dots, t_f - 1\}
\end{aligned}$$

State feedback control:

$$\begin{aligned}
u(t) = u(x(t)) &= \arg \min_{u \in U} \{ \underbrace{\ell(x(t), u) + V(t+1, \phi(x(t), u))}_{\text{computed using the measured state } x(t)} \} \\
x(t+1) &= \phi(x(t), u(t)) \qquad x(t_0) = x_0 \in X
\end{aligned}$$

example:

$$\begin{aligned}x(t) &\in X = \{x_1, x_2, x_4, x_5\} \\u(t) &\in U = \{u_1, u_2, u_3\} \\t &\in T = \{1, 2, 3\}\end{aligned}$$

$$\begin{aligned}\phi: X \times U &\rightarrow X & x(t+1) &= \phi(x(t), u(t)) \\ \ell: X \times U &\rightarrow \mathbb{R} & \ell(x(t), u(t)) & \\ \ell_f: X &\rightarrow \mathbb{R} & \ell_f(x(t_f)) &\end{aligned}$$

ϕ , ℓ and ℓ_f are given as follows:

$\phi(x_i, u_j)$	u_1	u_2	u_3	$\ell(x_i, u_j)$	u_1	u_2	u_3	$\ell_f(x_i)$
x_1	x_3	x_1	x_5	x_1	1	5	3	x_1
x_2	x_4	x_3	x_2	x_2	4	1	2	x_2
x_3	x_2	x_5	x_4	x_3	2	3	1	x_3
x_4	x_1	x_2	x_1	x_4	3	4	5	x_4
x_5	x_5	x_4	x_3	x_5	5	2	4	x_5

Let $x(1) \in X$ be given, and consider the optimal control problem:

$$\begin{aligned}J(1, x(1); u(\cdot)) &= \sum_{\tau=1}^2 \ell(x(\tau), u(\tau)) + \ell_f(x(3)) \\ &= \ell(x(1), u(1)) + \ell(x(2), u(2)) + \ell_f(x(3)) \\ &\inf_{u(1), u(2) \in U} J(1, x(1); u(\cdot))\end{aligned}$$

Bellman equation:

$$\begin{aligned}t = 3 : & \quad V(3, x) = \ell_f(x) & \text{for all } x \in X \\ t = 2 : & \quad V(2, x) = \inf_{u \in U} \{ \ell(x, u) + V(3, \phi(x, u)) \} & \text{for all } x \in X \\ t = 1 : & \quad V(1, x) = \inf_{u \in U} \{ \ell(x, u) + V(2, \phi(x, u)) \} & \text{for all } x \in X\end{aligned}$$

for $t = t_f = 3$:

$$V(3, x) = \ell_f(x)$$

for all $x \in X$

x_i	$\ell_f(x_i)$	x_i	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1	1	x_1			1
x_2	2	x_2			2
x_3	3	x_3			3
x_4	4	x_4			4
x_5	5	x_5			5

for $t = t_f - 1 = 2$:

$$V(2, x) = \inf_{u \in U} \{ \ell(x, u) + V(3, \phi(x, u)) \} \quad \text{for all } x \in X$$

$$V(2, x_1) = \inf_{u \in U} \left\{ \underbrace{\ell(x_1, u)}_{\begin{smallmatrix} 1 \\ 5 \\ 3 \end{smallmatrix}} + \underbrace{V(3, \phi(x_1, u))}_{\begin{smallmatrix} 3 \\ 1 \\ 5 \end{smallmatrix}} \right\} = 4$$

$$V(2, x_2) = \inf_{u \in U} \left\{ \underbrace{\ell(x_2, u)}_{\begin{smallmatrix} 4 \\ 1 \\ 2 \end{smallmatrix}} + \underbrace{V(3, \phi(x_2, u))}_{\begin{smallmatrix} 4 \\ 3 \\ 2 \end{smallmatrix}} \right\} = 4$$

$$V(2, x_3) = \inf_{u \in U} \left\{ \underbrace{\ell(x_3, u)}_{\begin{smallmatrix} 2 \\ 3 \\ 1 \end{smallmatrix}} + \underbrace{V(3, \phi(x_3, u))}_{\begin{smallmatrix} 2 \\ 5 \\ 4 \end{smallmatrix}} \right\} = 4$$

$$V(2, x_4) = \inf_{u \in U} \left\{ \underbrace{\ell(x_4, u)}_{\begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix}} + \underbrace{V(3, \phi(x_4, u))}_{\begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix}} \right\} = 4$$

$$V(2, x_5) = \inf_{u \in U} \left\{ \underbrace{\ell(x_5, u)}_{\begin{smallmatrix} 5 \\ 2 \\ 4 \end{smallmatrix}} + \underbrace{V(3, \phi(x_5, u))}_{\begin{smallmatrix} 5 \\ 4 \\ 3 \end{smallmatrix}} \right\} = 6$$

$\phi(x_i, u_j)$	u_1	u_2	u_3	$\ell(x_i, u_j)$	u_1	u_2	u_3	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1	x_3	x_1	x_5	x_1	1	5	3	x_1	4	1
x_2	x_4	x_3	x_2	x_2	4	1	2	x_2	4	2
x_3	x_2	x_5	x_4	x_3	2	3	1	x_3	4	3
x_4	x_1	x_2	x_1	x_4	3	4	5	x_4	4	4
x_5	x_5	x_4	x_3	x_5	5	2	4	x_5	6	5

for $t = 1$:

$$V(1, x) = \inf_{u \in U} \{\ell(x, u) + V(2, \phi(x, u))\} \quad \text{for all } x \in X$$

$$V(1, x_1) = \inf_{u \in U} \left\{ \underbrace{\ell(x_1, u)}_{\begin{smallmatrix} 1 \\ 5 \\ 3 \end{smallmatrix}} + \underbrace{V(2, \phi(x_1, u))}_{\begin{smallmatrix} 4 \\ 4 \\ 6 \end{smallmatrix}} \right\} = 5$$

$$V(1, x_2) = \inf_{u \in U} \left\{ \underbrace{\ell(x_2, u)}_{\begin{smallmatrix} 4 \\ 1 \\ 2 \end{smallmatrix}} + \underbrace{V(2, \phi(x_2, u))}_{\begin{smallmatrix} 4 \\ 4 \\ 4 \end{smallmatrix}} \right\} = 5$$

$$V(1, x_3) = \inf_{u \in U} \left\{ \underbrace{\ell(x_3, u)}_{\begin{smallmatrix} 2 \\ 3 \\ 1 \end{smallmatrix}} + \underbrace{V(2, \phi(x_3, u))}_{\begin{smallmatrix} 4 \\ 6 \\ 4 \end{smallmatrix}} \right\} = 5$$

$$V(1, x_4) = \inf_{u \in U} \left\{ \underbrace{\ell(x_4, u)}_{\begin{smallmatrix} 3 \\ 4 \\ 5 \end{smallmatrix}} + \underbrace{V(2, \phi(x_4, u))}_{\begin{smallmatrix} 4 \\ 4 \\ 4 \end{smallmatrix}} \right\} = 7$$

$$V(1, x_5) = \inf_{u \in U} \left\{ \underbrace{\ell(x_5, u)}_{\begin{smallmatrix} 5 \\ 2 \\ 4 \end{smallmatrix}} + \underbrace{V(2, \phi(x_5, u))}_{\begin{smallmatrix} 6 \\ 4 \\ 3 \end{smallmatrix}} \right\} = 6$$

$\phi(x_i, u_j)$	u_1	u_2	u_3	$\ell(x_i, u_j)$	u_1	u_2	u_3	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1	x_3	x_1	x_5	x_1	1	5	3	5	4	1
x_2	x_4	x_3	x_2	x_2	4	1	2	5	4	2
x_3	x_2	x_5	x_4	x_3	2	3	1	5	4	3
x_4	x_1	x_2	x_1	x_4	3	4	5	7	4	4
x_5	x_5	x_4	x_3	x_5	5	2	4	6	6	5

Let us consider the initial condition $x(1) = x_5$, then the optimal trajectory is given as follows:

$$\begin{aligned}
x(1) &= x_5 \\
u(1) &= \arg \min_{u \in U} \left\{ \underbrace{\ell(x_5, u)}_{\frac{5}{4}} + \underbrace{V(2, \phi(x_5, u))}_{\frac{6}{4}} \right\} && = u_2 \\
x(2) &= \phi(x_5, u_2) = x_4 \\
u(2) &= \arg \min_{u \in U} \left\{ \underbrace{\ell(x_4, u)}_{\frac{3}{4}} + \underbrace{V(3, \phi(x_4, u))}_{\frac{1}{1}} \right\} && = u_1 \\
x(3) &= \phi(x_4, u_1) = x_1
\end{aligned}$$

$$\begin{aligned}
\min_{u(1), u(2) \in U} J(1, x_5; u(\cdot)) &= \ell(x_5, u_2) + \ell(x_4, u_1) + \ell_f(x_1) \\
&= 2 + 3 + 1 = 6 = V(1, x_5)
\end{aligned}$$

$\phi(x_i, u_j)$	u_1	u_2	u_3	$\ell(x_i, u_j)$	u_1	u_2	u_3	$\ell_f(x_i)$	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$		
x_1	x_3	x_1	x_5	x_1	1	5	3	x_1	1	x_1	5	4	1
x_2	x_4	x_3	x_2	x_2	4	1	2	x_2	2	x_2	5	4	2
x_3	x_2	x_5	x_4	x_3	2	3	1	x_3	3	x_3	5	4	3
x_4	x_1	x_2	x_1	x_4	3	4	5	x_4	4	x_4	7	4	4
x_5	x_5	x_4	x_3	x_5	5	2	4	x_5	5	x_5	6	6	5