$$x(t+1) = f(x(t), u(t)) \qquad x(t_0) = x_0 \qquad t \in T = \{t_0, t_0+1, \ldots, t_{\mathrm{f}}\}$$
$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

$$\begin{aligned} f &: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n & & x(t+1) = f(x(t), u(t)) \\ \ell &: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} & & \ell(x(t), u(t)) \\ \ell_{\mathrm{f}} &: \mathbb{R}^n \to \mathbb{R} & & \ell_{\mathrm{f}}(x(t_{\mathrm{f}})) \end{aligned}$$

Let $x(t_0) = x_0 \in \mathbb{R}^n$ be given, and consider the optimal control problem:

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_{\mathrm{f}}-1} \ell(x(\tau), u(\tau)) + \ell_{\mathrm{f}}(x(t_{\mathrm{f}}))$$

$$\inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in T}} J(t_0, x_0; u(\cdot))$$

Define the cost-to-go:

$$V : T \times \mathbb{R}^n \to \mathbb{R} \qquad V(t, x) = \inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in\{t, t+1, \ldots, t_{\mathrm{f}}\}}} J(t, x; u(\cdot))$$

Note that computing the cost-to-go $V(t_0, x_0)$ from the initial state x_0 at the initial time t_0 essentially amounts to minimize the cost $J(t_0, x_0; u(\cdot))$.

If $t = t_\mathrm{f}$:

$$\begin{aligned} V(t_\mathrm{f}, x) &= \inf_{u(t_\mathrm{f}) \in \mathbb{R}^m} J(t_\mathrm{f}, x; u(t_\mathrm{f})) \\ &= \inf_{u(t_\mathrm{f}) \in \mathbb{R}^m} \underbrace{\ell_\mathrm{f}(x)}_{\substack{\text{independent} \\ \text{of } u(t_\mathrm{f})}} = \ell_\mathrm{f}(x) \end{aligned}$$

If $t = t_\mathrm{f} - 1$:

$$\begin{aligned} V(t_\mathrm{f} - 1, x) &= \inf_{u(t_\mathrm{f}-1), u(t_\mathrm{f}) \in \mathbb{R}^m} J(t_\mathrm{f} - 1, x; u(\cdot)) \\ &= \inf_{u(t_\mathrm{f}-1), u(t_\mathrm{f}) \in \mathbb{R}^m} \{\underbrace{\ell(x(t_\mathrm{f} - 1), u(t_\mathrm{f} - 1))}_{\text{independent of } u(t_\mathrm{f})} + \underbrace{\ell_\mathrm{f}(x(t_\mathrm{f}))}_{\substack{\text{depend on both} \\ u(t_\mathrm{f}-1) \text{ and } u(t_\mathrm{f})}} \} \\ &= \inf_{u(t_\mathrm{f}-1) \in \mathbb{R}^m} \{\ell(x, u(t_\mathrm{f} - 1)) + \underbrace{\inf_{u(t_\mathrm{f}) \in \mathbb{R}^m} \ell_\mathrm{f}(x(t_\mathrm{f}))}_{= V(t_\mathrm{f}, x(t_\mathrm{f})) = V(t_\mathrm{f}, f(x, u(t_\mathrm{f} - 1)))} \} \\ &= \inf_{u(t_\mathrm{f}-1) \in \mathbb{R}^m} \{\ell(x, u(t_\mathrm{f} - 1)) + V(t_\mathrm{f}, f(x, u(t_f - 1)))\} \\ &= \inf_{u \in \mathbb{R}^m} \{\ell(x, u) + V(t_\mathrm{f}, f(x, u))\} \end{aligned}$$

For $t < t_\mathrm{f}$:

$$\begin{aligned} V(t, x) &= \inf_{\substack{u(\tau) \in \mathbb{R}^m \\ \tau \in \{t, t+1, \ldots, t_\mathrm{f}\}}} J(t, x; u(\cdot)) \\ &= \inf_{\substack{u(\tau) \in \mathbb{R}^m \\ \tau \in \{t, t+1, \ldots, t_\mathrm{f}\}}} \{\sum_{\tau = t}^{t_\mathrm{f} - 1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))\} \\ &= \inf_{\substack{u(\tau) \in \mathbb{R}^m \\ \tau \in \{t, t+1, \ldots, t_\mathrm{f}\}}} \{ \underbrace{\ell(x, u(t))}_{\substack{\text{independent of } u(\tau), \\ \tau \in \{t+1, t+2, \ldots, t_\mathrm{f}\}}} + \underbrace{\sum_{\tau = t+1}^{t_\mathrm{f} - 1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))}_{\text{depend on all } u(\tau),\ \tau \in \{t, t+1, \ldots, t_\mathrm{f}\}} \} \\ &= \inf_{u(t) \in \mathbb{R}^m} \{\ell(x, u(t)) + \underbrace{\inf_{\substack{u(\tau) \in \mathbb{R}^m \\ \tau \in \{t+1, t+2, \ldots, t_\mathrm{f}\}}} \{\sum_{\tau = t+1}^{t_\mathrm{f} - 1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))\}}_{= V(t+1, x(t+1)) = V(t+1, f(x, u(t)))} \} \\ &= \inf_{u(t) \in \mathbb{R}^m} \{\ell(x, u(t)) + V(t+1, f(x, u(t)))\} \\ &= \inf_{u \in \mathbb{R}^m} \{\ell(x, u) + V(t+1, f(x, u))\} \end{aligned}$$

Bellman equation:

$$\begin{aligned} V(t_\mathrm{f}, x) &= \ell_\mathrm{f}(x) && \text{for all } x \in \mathbb{R}^n \\ V(t, x) &= \inf_{u\in\mathbb{R}^m} \{\ell(x,u) + V(t+1, f(x,u))\} && \text{for all } x \in \mathbb{R}^n \text{ and all } t \in \{t_0, t_{0+1}, \ldots, t_\mathrm{f}-1\} \end{aligned}$$

Let us suppose that the cost-to-go V has been determined. For a given state x at time t, the optimal input $u(t)$ is given as

$$u(t) = \arg\min_{u\in\mathbb{R}^m} \{\ell(x,u) + V(t+1, f(x,u))\}$$

This inspires the implantation of the optimal control in a state feed back form:

$$\begin{aligned} u(t) = u(x(t)) &= \arg\min_{u\in\mathbb{R}^m} \{ \underbrace{\ell(x(t),u) + V(t+1, f(x(t),u))}_{\text{computed using the measured state } x(t)} \} \\ x(t+1) &= f(x(t), u(t)) \qquad x(t_0) = x_0 \end{aligned}$$