$$\dot{x}(t) = f(x(t), u(t)) \qquad x(t_0) = x_0 \qquad t \in [\, t_0, t_\mathrm{f} \,]$$
$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

$$\begin{aligned} f &: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n & \qquad & \dot{x}(t) = f(x(t), u(t)) \\ \ell &: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} & & \ell(x(t), u(t)) \\ \ell_\mathrm{f} &: \mathbb{R}^n \to \mathbb{R} & & \ell_\mathrm{f}(x(t_\mathrm{f})) \end{aligned}$$

Let $x(t_0) = x_0$ be given, and consider the optimal control problem:

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_\mathrm{f}} \ell(x(\tau), u(\tau)) d\tau + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

$$\inf_{\substack{u(\tau) \\ \tau \in [\, t_0, t_\mathrm{f} \,]}} J(t_0, x_0; u(\cdot))$$

Define the cost-to-go:

$$V:\ \mathbb{R}\times\mathbb{R}^n \to \mathbb{R} \qquad\qquad V(t,x) = \inf_{\substack{u(\tau)\\ \tau\in[\,t,t_\mathrm{f}\,]}} J(t,x;u(\cdot))$$

Note that computing the cost-to-go $V(t_0,x_0)$ from the initial state x_0 at the initial time t_0 essentially amounts to minimize the cost $J(t_0,x_0;u(\cdot))$.

If $t = t_\mathrm{f}$:

$$\begin{aligned} V(t_\mathrm{f}, x) &= \inf_{u(t_\mathrm{f})} J(t_\mathrm{f}, x; u(t_\mathrm{f})) \\ &= \inf_{u(t_\mathrm{f})} \underbrace{\ell_\mathrm{f}(x)}_{\substack{\text{independent} \\ \text{of } u(t_\mathrm{f})}} = \ell_\mathrm{f}(x) \end{aligned}$$

For $t < t_\mathrm{f}$, let us pick some small positive constant δt so that $t + \delta t$ is still smaller than t_f:

$$
\begin{aligned}
V(t,x) &= \inf_{\substack{u(\tau)\\ \tau\in[\,t,t_\mathrm{f}\,]}} J(t,x;u(\cdot)) \\
&= \inf_{\substack{u(\tau)\\ \tau\in[\,t,t_\mathrm{f}\,]}} \{\int_t^{t_\mathrm{f}} \ell(x(\tau),u(\tau))d\tau + \ell_\mathrm{f}(x(t_\mathrm{f}))\} \\
&= \inf_{\substack{u(\tau)\\ \tau\in[\,t,t_\mathrm{f}\,]}} \{\underbrace{\int_t^{t+\delta t} \ell(x(\tau),u(\tau))d\tau}_{\substack{\text{independent of } u(\tau),\\ \tau\in[\,t+\delta t,t_\mathrm{f}\,]}} + \underbrace{\int_{t+\delta t}^{t_\mathrm{f}} \ell(x(\tau),u(\tau))d\tau + \ell_\mathrm{f}(x(t_\mathrm{f}))\}}_{\text{depend on all } u(\tau),\ \tau\in[\,t,t_\mathrm{f}\,]} \\
&= \inf_{\substack{u(\tau)\\ \tau\in[\,t,t+\delta t\,]}} \{\int_t^{t+\delta t} \ell(x(\tau),u(\tau))d\tau + \underbrace{\inf_{\substack{u(\tau)\\ \tau\in[\,t+\delta t,t_\mathrm{f}\,]}} \{\int_{t+\delta t}^{t_\mathrm{f}} \ell(x(\tau),u(\tau))d\tau + \ell_\mathrm{f}(x(t_\mathrm{f}))\}\}}_{=V(t+\delta t,\,x(t+\delta t))} \\
&= \inf_{\substack{u(\tau)\\ \tau\in[\,t,t+\delta t\,]}} \{\int_t^{t+\delta t} \ell(x(\tau),u(\tau))d\tau + V(t+\delta t, x(t+\delta t))\} \\
V(t,x) &= \inf_{\substack{u(\tau)\\ \tau\in[\,t,t+\delta t\,]}} \{\int_t^{t+\delta t} \ell(x(\tau),u(\tau))d\tau + V(t+\delta t, x(t+\delta t))\}
\end{aligned}
$$

This represents the principle of optimality.

We now consider the infinitesimal version of the above relationship. Let us consider first-order Taylor expansion:

$$\begin{aligned} x(t+\delta t) &= x(t) + \delta x \\ &= x(t) + \frac{dx}{dt}(t)\delta t + o(\delta t) \\ &= x(t) + f(x(t), u(t))\delta t + o(\delta t) \\ &= x + \underbrace{f(x, u(t))\delta t + o(\delta t)}_{\delta x} \end{aligned}$$

This allows us the expressions:

$$\begin{aligned} V(t+\delta t, x(t+\delta t)) &= V(t, x(t)) + \frac{\partial V}{\partial t}(t, x(t))\delta t + o(\delta t) + \left(\frac{\partial V}{\partial x}(t, x(t))\right)^{\mathrm{T}} \delta x + o(\delta x) \\ &= V(t, x) + \frac{\partial V}{\partial t}(t, x)\delta t + \left(\frac{\partial V}{\partial x}(t, x)\right)^{\mathrm{T}} f(x, u(t))\delta t + o(\delta t) \end{aligned}$$

and

$$\begin{aligned} \int_t^{t+\delta t} \ell(x(\tau), u(\tau)) d\tau &= \int_0^{\delta t} \ell(x(t+\tau), u(t+\tau)) d\tau \\ &= \ell(x(t), u(t))\delta t + o(\delta t) \\ &= \ell(x, u(t))\delta t + o(\delta t) \end{aligned}$$

By substituting the above expressions, we obtain

$$\begin{aligned} V(t, x) &= \inf_{\substack{u(\tau) \\ \tau \in [\, t, t+\delta t\,]}} \{\ell(x, u(t))\delta t + V(t, x) + \frac{\partial V}{\partial t}(t, x)\delta t + \left(\frac{\partial V}{\partial x}(t, x)\right)^{\mathrm{T}} f(x, u(t))\delta t + o(\delta t)\} \\ 0 &= \frac{\partial V}{\partial t}(t, x)\delta t + \inf_{\substack{u(\tau) \\ \tau \in [\, t, t+\delta t\,]}} \{\ell(x, u(t))\delta t + \left(\frac{\partial V}{\partial x}(t, x)\right)^{\mathrm{T}} f(x, u(t))\delta t + o(\delta t)\} + V(t, x) - V(t, x) \\ &= \frac{\partial V}{\partial t}(t, x)\delta t + \inf_{\substack{u(\tau) \\ \tau \in [\, t, t+\delta t\,]}} \{\ell(x, u(t))\delta t + \left(\frac{\partial V}{\partial x}(t, x)\right)^{\mathrm{T}} f(x, u(t))\delta t + o(\delta t)\} \end{aligned}$$

We note here that the values $u(\tau)$, $\tau \in (\, t, t+\delta t\,]$ affect the expression inside the infimum only through the $o(\delta t)$ term. Let us now divide by δt, we have

$$0 = \frac{\partial V}{\partial t}(t, x) + \inf_{\substack{u(\tau) \\ \tau \in [\, t, t+\delta t\,]}} \{\ell(x, u(t)) + \left(\frac{\partial V}{\partial x}(t, x)\right)^{\mathrm{T}} f(x, u(t)) + \frac{o(\delta t)}{\delta t}\}$$

Let us take δt to be small as $\delta t \to 0$. The higher order term $o(\delta t)/\delta t$ disappears, and the infimum is taken over the instantaneous value of u at time t. We obtain

$$0 = \frac{\partial V}{\partial t}(t, x) + \inf_{u \in \mathbb{R}^m} \{\ell(x, u) + \left(\frac{\partial V}{\partial x}(t, x)\right)^{\mathrm{T}} f(x, u)\}$$

Hamilton-Jacobi-Bellman equation:

$$V(t_{\mathrm{f}}, x) = \ell_{\mathrm{f}}(x) \qquad \text{for all } x \in \mathbb{R}^n$$

$$0 = \frac{\partial V}{\partial t}(t,x) + \inf_{u \in \mathbb{R}^m} \{\ell(x,u) + \left(\frac{\partial V}{\partial x}(t,x)\right)^{\mathrm{T}} f(x,u)\} \qquad \text{for all } x \in \mathbb{R}^n \text{ and all } t \in [\, t_0, t_{\mathrm{f}}\,)$$

Let us suppose that the cost-to-go $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ has been determined. For a given state x at time t, the optimal input $u(t)$ is given as

$$u(t) = \arg\min_{u \in \mathbb{R}^m} \{\ell(x,u) + \left(\frac{\partial V}{\partial x}(t,x)\right)^{\mathrm{T}} f(x,u)\}$$

This inspires the implementation of the optimal control in a state feedback form:

$$\begin{aligned} u(t) &= u(x(t)) \\ &= \arg\min_{u \in \mathbb{R}^m} \underbrace{\{\ell(x(t),u) + \left(\frac{\partial V}{\partial x}(t,x(t))\right)^{\mathrm{T}} f(x(t),u)\}}_{\text{computed using the measured state } x(t)} \\ \dot{x}(t) &= f(x(t),u(t)) \qquad x(t_0) = x_0 \end{aligned}$$