Advanced Control Systems Engineering I:

Optimal Control

dynamical systems

$$\dot{x}(t) = f(x(t), u(t))$$

dynamical systems

$$\dot{x}(t) = f(x(t), u(t)) \qquad x(t_0) = x_0$$

dynamical systems

$$\dot{x}(t) = f(x(t), u(t)) \qquad x(t_0) = x_0 \qquad t \in [\, t_0, t_\mathrm{f} \,]$$

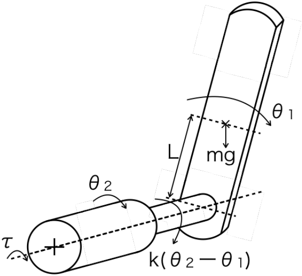
dynamical systems

$$\dot{x}(t) = f(x(t), u(t)) \qquad x(t_0) = x_0 \qquad t \in [\, t_0, t_\mathrm{f} \,]$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

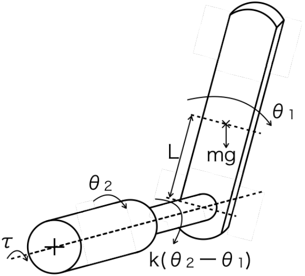
robotic manipulator

dynamical systems



robotic manipulator

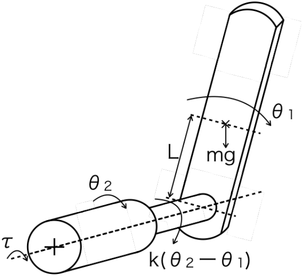
dynamical systems



$$J_1\ddot{\theta}_1(t) = k(\theta_2(t) - \theta_1(t)) + mgL\sin\theta_1(t)$$
$$J_2\ddot{\theta}_2 = \tau(t) - k(\theta_2(t) - \theta_1(t))$$

robotic manipulator

dynamical systems

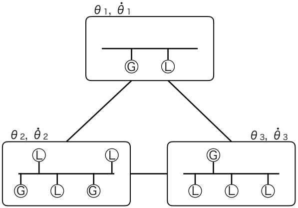


$$J_1\ddot{\theta}_1(t) = k(\theta_2(t) - \theta_1(t)) + mgL\sin\theta_1(t)$$
$$J_2\ddot{\theta}_2 = \tau(t) - k(\theta_2(t) - \theta_1(t))$$

$$\dot{x}(t) = f(x(t), u(t)) \qquad x = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \qquad u = \tau$$

swing equation

dynamical systems



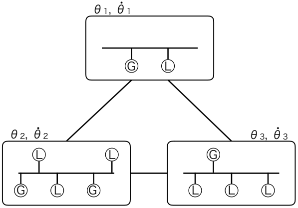
[J-Power, http://www.jpower.co.jp]

n areas $\quad N = \{1, 2, \ldots, n\}$

neighbors of area $i \quad N_i \subset N$

swing equation

dynamical systems



[J-Power, http://www.jpower.co.jp]

n areas $\quad N = \{1, 2, \ldots, n\}$ $\qquad$ neighbors of area i $\quad N_i \subset N$

$$H_i \ddot{\theta}_i(t) = \sum_{j=1}^{n_i^{\mathrm{g}}} P_{ij}^{\mathrm{g}}(t) - \sum_{j=1}^{n_i^{\mathrm{l}}} P_{ij}^{\mathrm{l}}(t) - \sum_{j \in N_i} \frac{1}{X_{ij}} (\theta_i(t) - \theta_j(t)) \qquad i \in N$$

$$P_{ij}^{\mathrm{g}} = C_{ij} x_{ij}(t) \qquad \dot{x}_{ij}(t) = A_{ij} x_{ij}(t) + B_{ij} u_{ij}(t) \quad j = 1, 2, \ldots, n_i^{\mathrm{g}}$$

optimal control problem

$$\dot{x}(t) = f(x(t), u(t)) \qquad x(t_0) = x_0 \qquad t \in [\, t_0, t_\mathrm{f} \,]$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

optimal control problem

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$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_{\mathrm{f}}} \ell(x(\tau), u(\tau))\, d\tau + \ell_{\mathrm{f}}(x(t_{\mathrm{f}}))$$

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$$\inf_{u(\cdot)} J(t_0, x_0; u(\cdot))$$

HJB equation

optimal control problem

$$\dot{x}(t) = f(x(t), u(t)) \qquad x(t_0) = x_0 \qquad t \in [\, t_0, t_\mathrm{f} \,]$$

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$$0 = \frac{\partial V}{\partial t}(t, x) + \inf_{u \in \mathbb{R}^m} \left\{ \ell(x, u) + \left(\frac{\partial V}{\partial x}(t, x) \right)^{\mathrm{T}} f(x, u) \right\}$$

$$V(t_\mathrm{f}, x) = \ell_\mathrm{f}(x) \qquad \text{for all } x \in \mathbb{R}^n$$

HJB equation

optimal control problem

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$$V(t_\mathrm{f}, x) = \ell_\mathrm{f}(x) \qquad \text{for all } x \in \mathbb{R}^n$$

$$u^*(t) = \arg\min_{u \in \mathbb{R}^m} \left\{ \ell(x(t), u) + \left(\frac{\partial V}{\partial x}(t, x(t)) \right)^\mathrm{T} f(x(t), u) \right\}$$

maximum principle

optimal control problem

$$\dot{x}(t) = f(x(t), u(t)) \qquad x(t_0) = x_0 \qquad t \in [\, t_0, t_\mathrm{f} \,]$$

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$$\inf_{u(\cdot)} J(t_0, x_0; u(\cdot))$$

$$H(x, u, p) = \ell(x, u) + p^{\mathrm{T}} f(x, u)$$

maximum principle

optimal control problem

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$$\inf_{u(\cdot)} J(t_0, x_0; u(\cdot))$$

$$H(x, u, p) = \ell(x, u) + p^\mathrm{T} f(x, u)$$

$$\dot{p}(t) = -\frac{\partial H}{\partial x}(x(t), u(t), p(t)) \qquad p(t_\mathrm{f}) = \frac{\partial \ell_\mathrm{f}}{\partial x}(x(t_\mathrm{f}))$$

$$u(t) = \arg\min_{u \in \mathbb{R}^m} H(x(t), u, p(t))$$

maximum principle

optimal control problem

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$$u(t) = \arg \min_{u \in \mathbb{R}^m} H(x(t), u, p(t))$$

- maximum principle
- calculus of variations

linear systems

optimal control problem

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x(t_0) = x_0 \qquad t \geq 0$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

linear systems

optimal control problem

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x(t_0) = x_0 \qquad t \geq 0$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

$$J(t_0, x_0; u(\cdot)) = \int_0^\infty x^{\mathrm{T}}(\tau) R x(t) + u^{\mathrm{T}}(\tau) Q u(\tau) dt$$

$$\inf_{u(\cdot)} J(t_0, x_0; u(\cdot))$$

linear systems

optimal control problem

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x(t_0) = x_0 \qquad t \geq 0$$

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$$\inf_{u(\cdot)} J(t_0, x_0; u(\cdot))$$

$$0 = PA + A^{\mathrm{T}}P - PBR^{-1}B^{\mathrm{T}}P + Q$$

$$u^*(t) = -\frac{1}{2} R^{-1} B^{\mathrm{T}} P x(t)$$

optimal control problem

$$\dot{x}(t) = f(x(t), u(t)) \qquad x(t_0) = x_0 \qquad t \in [\, t_0, t_\mathrm{f} \,]$$

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$$0 = \frac{\partial V}{\partial t}(t, x) + \inf_{u \in \mathbb{R}^m} \left\{ \ell(x, u) + \left(\frac{\partial V}{\partial x}(t, x) \right)^\mathrm{T} f(x, u) \right\}$$

$$V(t_\mathrm{f}, x) = \ell_\mathrm{f}(x) \qquad \text{for all } x \in \mathbb{R}^n$$

$$u^*(t) = \arg\min_{u \in \mathbb{R}^m} \left\{ \ell(x(t), u) + \left(\frac{\partial V}{\partial x}(t, x(t)) \right)^\mathrm{T} f(x(t), u) \right\}$$

optimal control problem

- dynamic programming

$$0 = \frac{\partial V}{\partial t}(t, x) + \inf_{u \in \mathbb{R}^m} \{\ell(x, u) + \left(\frac{\partial V}{\partial x}(t, x)\right)^{\mathrm{T}} f(x, u)\}$$

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optimal control problem

- dynamic programming
  - multistage decision process

$$0 = \frac{\partial V}{\partial t}(t, x) + \inf_{u \in \mathbb{R}^m} \left\{ \ell(x, u) + \left(\frac{\partial V}{\partial x}(t, x) \right)^{\mathrm{T}} f(x, u) \right\}$$

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optimal control problem

- dynamic programming
  - multistage decision process
  - Hamilton-Jacobi-Bellman equation

$$0 = \frac{\partial V}{\partial t}(t,x) + \inf_{u \in \mathbb{R}^m} \left\{ \ell(x,u) + \left(\frac{\partial V}{\partial x}(t,x) \right)^{\mathrm{T}} f(x,u) \right\}$$

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optimal control problem

- dynamic programming
  - multistage decision process
  - Hamilton-Jacobi-Bellman equation
- principle of optimality

$$0 = \frac{\partial V}{\partial t}(t, x) + \inf_{u \in \mathbb{R}^m} \left\{ \ell(x, u) + \left(\frac{\partial V}{\partial x}(t, x) \right)^{\mathrm{T}} f(x, u) \right\}$$

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- dynamic programming
- principle of optimality
- optimal control of finite state systems
- optimal control of discrete-time systems
- optimal control of continuous-time systems
- optimal control of linear systems
- decentralized optimal control
  - decentralization and integration via mechanism design

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continuous-time systems

optimal control problem

$$\dot{x}(t) = f(x(t), u(t)) \qquad x(t_0) = x_0 \qquad t \in [\, t_0, t_\mathrm{f} \,]$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_\mathrm{f}} \ell(x(\tau), u(\tau))\, d\tau + \ell_\mathrm{f}(x_\mathrm{f}(t_\mathrm{f}))$$

$$\inf_{u(\cdot)} J(t_0, x_0; u(\cdot))$$

continuous-time systems

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continuous-time systems

optimal control problem

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discrete-time systems

optimal control problem

$$x(t+1) = f(x(t), u(t)) \qquad x(t_0) = x_0 \qquad t \in [\, t_0, t_0+1, \ldots, t_\mathrm{f} \,]$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

discrete-time systems

optimal control problem

$$x(t+1) = f(x(t), u(t)) \qquad x(t_0) = x_0 \qquad t \in [\, t_0, t_0 + 1, \ldots, t_{\mathrm{f}} \,]$$

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$$J(t_0, x_0; u(\cdot)) = \sum_{\tau = t_0}^{t_{\mathrm{f}}-1} \ell(x(\tau), u(\tau)) + \ell_{\mathrm{f}}(x(t_f))$$

discrete-time systems

optimal control problem

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discrete-time systems

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$$V(t,x) = \inf_{u \in \mathbb{R}^m} \{\ell(x,u) + V(t+1, f(x,u))\}$$

$$V(t_\mathrm{f}, x) = \ell_\mathrm{f}(x) \qquad \text{for all } x \in \mathbb{R}^n$$

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optimal control problem

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finite state systems

optimal control problem

$$x(t) \in X = \{x_1, x_2, \ldots, x_n\} \qquad u(t) \in U = \{u_1, u_2, \ldots, u_m\}$$

$$t \in [\, t_0, t_0 + 1, \ldots, t_\mathrm{f} \,]$$

finite state systems

optimal control problem

$$x(t) \in X = \{x_1, x_2, \ldots, x_n\} \qquad u(t) \in U = \{u_1, u_2, \ldots, u_m\}$$

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finite state systems

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$$\inf_{u(\cdot)} J(t_0, x_0; u(\cdot))$$

finite state systems

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$$\inf_{u(\cdot)} J(t_0, x_0; u(\cdot))$$

$x(t+1)$	u_1	u_2	$\cdots$	u_m
x_1	x_3	x_{n-2}	$\cdots$	x_1
x_2	x_2	x_8	$\cdots$	x_n
$\vdots$				
x_n	x_5	x_{n-7}	$\cdots$	x_2

finite state systems

optimal control problem

$$x(t) \in X = \{x_1, x_2, \ldots, x_n\} \qquad u(t) \in U = \{u_1, u_2, \ldots, u_m\}$$

$$t \in [\, t_0, t_0 + 1, \ldots, t_\mathrm{f} \,]$$

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

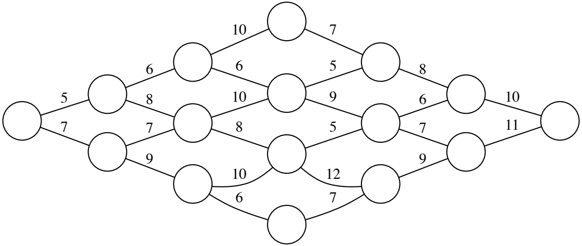
$$\inf_{u(\cdot)} J(t_0, x_0; u(\cdot))$$

$\ell(x_i, u_j)$	u_1	u_2	$\cdots$	u_m
x_1	3	2	$\cdots$	-1
x_2	2	-2	$\cdots$	6
$\vdots$				
x_n	-1	5	$\cdots$	1.2

	$\ell_\mathrm{f}(x_i)$
x_1	3
x_2	2
$\vdots$	
x_n	-1

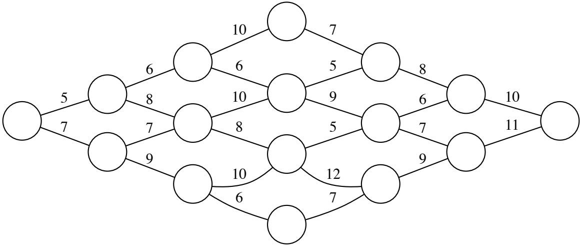
finite state systems

optimal control problem



finite state systems

optimal control problem

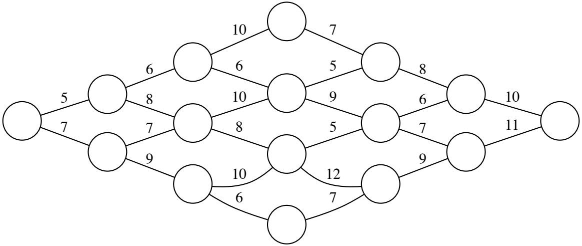


$$X = \{x_1, x_2, \ldots, x_{16}\} \qquad u(t) \in U = \{u_{\text{u}}, u_{\text{d}}\}$$

$$t \in [\,0, 1, 2, 3, 4, 5, 6\,]$$

finite state systems

optimal control problem



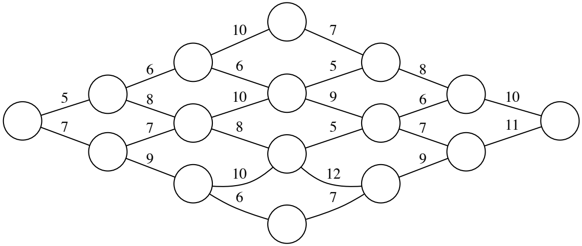
$$X = \{x_1, x_2, \ldots, x_{16}\} \qquad u(t) \in U = \{u_\mathrm{u}, u_\mathrm{d}\}$$

$$t \in [\,0, 1, 2, 3, 4, 5, 6\,]$$

$$\inf_{u(\cdot)} J(0, x_1; u(\cdot)) = \inf_{u(\cdot)} \sum_{\tau=0}^{6} \ell(x(\tau), u(\tau))$$

finite state systems

optimal control problem



$$X = \{x_1, x_2, \ldots, x_{16}\} \qquad u(t) \in U = \{u_\mathrm{u}, u_\mathrm{d}\}$$

$$t \in [\,0, 1, 2, 3, 4, 5, 6\,]$$

$$\inf_{u(\cdot)} J(0, x_1; u(\cdot)) = \inf_{u(\cdot)} \sum_{\tau=0}^{6} \ell(x(\tau), u(\tau))$$

minimum-cost path problem

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dynamic programming

DYNAMIC

PROGRAMMING

BY

RICHARD BELLMAN

In his 1957 book, R. E. Bellman wrote:

DYNAMIC PROGRAMMING

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In his 1957 book, R. E. Bellman wrote:

stifles analysis and greatly impedes computation?

In order to answer this, let us turn to the previously enunciated principle that it is the *structure* of the policy which is essential. What does this mean precisely? It means that we wish to know the characteristics of the system which determine the decision to be made at any particular stage of the process. Put another way, in place of determining the optimal sequence of decisions from some *fixed* state of the system, we wish to determine the optimal decision to be made at *any* state of the system. Only if we know the latter, do we understand the intrinsic structure of the solution.

The mathematical advantage of this formulation lies first of all in

principle of optimality

[Bellman, 1957, p. 83]

principle of optimality

§ 3. The principle of optimality

In each process, the functional equation governing the process was obtained by an application of the following intuitive:

PRINCIPLE OF OPTIMALITY. *An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.*

The mathematical transliteration of this simple principle will yield all the functional equations we shall encounter throughout the remainder of the book. A proof by contradiction is immediate.

[Bellman, 1957, p. 83]

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linear systems

optimal control problem

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x(t_0) = x_0 \qquad t \geq 0$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

linear systems

optimal control problem

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x(t_0) = x_0 \qquad t \geq 0$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

$$J(t_0, x_0; u(\cdot)) = \int_0^\infty x^{\mathrm{T}}(\tau) R x(t) + u^{\mathrm{T}}(\tau) Q u(\tau) dt$$

$$\inf_{u(\cdot)} J(t_0, x_0; u(\cdot))$$

linear systems

optimal control problem

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x(t_0) = x_0 \qquad t \geq 0$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

$$J(t_0, x_0; u(\cdot)) = \int_0^\infty x^{\mathrm{T}}(\tau) R x(t) + u^{\mathrm{T}}(\tau) Q u(\tau) \, dt$$

$$\inf_{u(\cdot)} J(t_0, x_0; u(\cdot))$$

$$0 = PA + A^{\mathrm{T}} P - PBR^{-1} B^{\mathrm{T}} P + Q$$

$$u^*(t) = -\frac{1}{2} R^{-1} B^{\mathrm{T}} P x(t)$$

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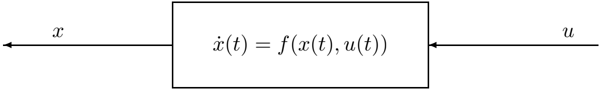
nonlinear dynamical systems and linear approximations

$$\dot{x}(t) = f(x(t), u(t))$$

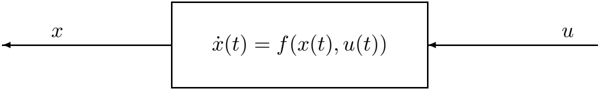
nonlinear dynamical systems and linear approximations

$$\dot{x}(t) = f(x(t), u(t))$$



nonlinear dynamical systems and linear approximations

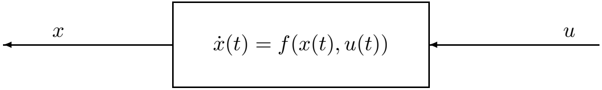
$$\dot{x}(t) = f(x(t), u(t))$$



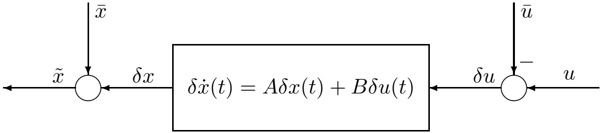
$$\delta \dot{x}(t) = A \delta x(t) + B \delta u(t)$$

nonlinear dynamical systems and linear approximations

$$\dot{x}(t) = f(x(t), u(t))$$



$$\delta\dot{x}(t) = A\delta x(t) + B\delta u(t)$$



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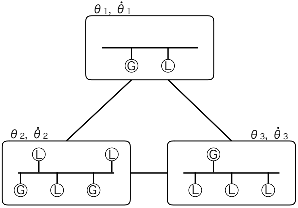
swing equation

dynamical systems



[J-Power, `http://www.jpower.co.jp`]

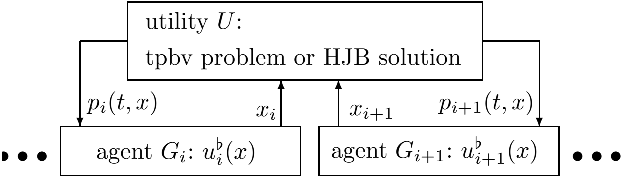
n areas $\quad N = \{1, 2, \ldots, n\}$ neighbors of area $i \quad N_i \subset N$

$$H_i \ddot\theta_i(t) = \sum_{j=1}^{n_i^{\mathrm{g}}} P_{ij}^{\mathrm{g}}(t) - \sum_{j=1}^{n_i^{\mathrm{l}}} P_{ij}^{\mathrm{l}}(t) - \sum_{j \in N_i} \frac{1}{X_{ij}} (\theta_i(t) - \theta_j(t)) \qquad i \in N$$

$$P_{ij}^{\mathrm{g}} = C_{ij} x_{ij}(t) \qquad \dot x_{ij}(t) = A_{ij} x_{ij}(t) + B_{ij} u_{ij}(t) \quad j = 1, 2, \ldots, n_i^{\mathrm{g}}$$

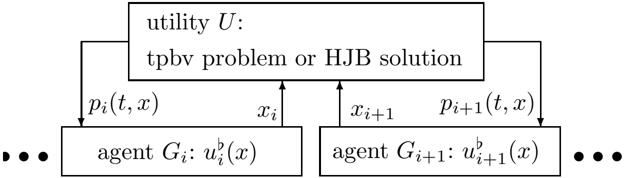
decentralization and integration via mechanism design

decentralized optimal control



decentralization and integration via mechanism design

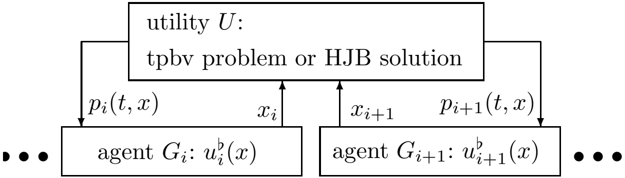
decentralized optimal control



- sets (virtual) prices p_i to G_i $\implies$ $u_i^* = u_i^\flat$
- receives messages x_i from G_i

decentralization and integration via mechanism design

decentralized optimal control



- sets (virtual) prices p_i to $G_i \implies u_i^* = u_i^\flat$
- receives messages x_i from G_i
- strategic behavior of individuals G_is could result in a wrong decision
- social mechanism: strategic behavior does not make any profit

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