

# Advanced Control Systems Engineering I:

## Optimal Control

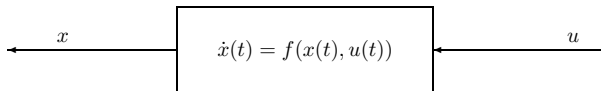
# contents

## optimal control

- ▶ nonlinear dynamical systems and linear approximations
- ▶ dynamic programming
- ▶ the principle of optimality
- ▶ optimal control of finite state systems
- ▶ optimal control of discrete-time systems
- ▶ optimal control of continuous-time systems
- ▶ optimal control of linear systems
- ▶ decentralized optimal control
  - ▶ decentralization and integration via mechanism design

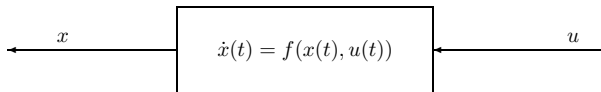
# nonlinear dynamical systems and linear approximations

$$\dot{x}(t) = f(x(t), u(t)) \quad x(t) \in \mathbb{R}^n \quad u(t) \in \mathbb{R}^m$$

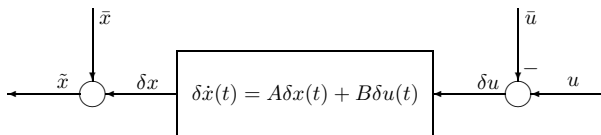


# nonlinear dynamical systems and linear approximations

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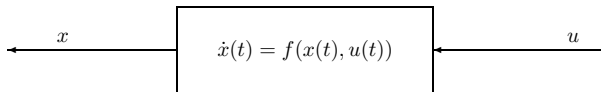


$$\delta\dot{x}(t) = A\delta x(t) + B\delta u(t) \quad A \in \mathbb{R}^{n \times n} \quad B \in \mathbb{R}^{n \times m}$$

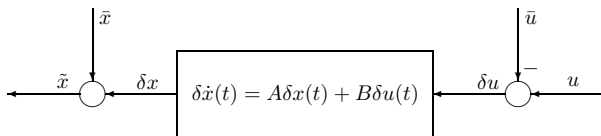


# nonlinear dynamical systems and linear approximations

$$\dot{x}(t) = f(x(t), u(t)) \quad x(t) \in \mathbb{R}^n \quad u(t) \in \mathbb{R}^m$$



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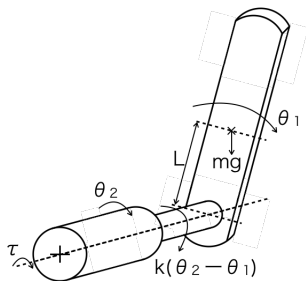
$$A = \left( \frac{\partial f}{\partial x}(\bar{x}, \bar{u}) \right)^T \quad B = \left( \frac{\partial f}{\partial u}(\bar{x}, \bar{u}) \right)^T$$

# dynamical systems

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) & x(t_0) &= x_0 & t &\in [t_0, t_f] \\ & & x(t) &\in \mathbb{R}^n & u(t) &\in \mathbb{R}^m \end{aligned}$$

# robotic manipulator

## dynamical systems



$$J_1 \ddot{\theta}_1(t) = k(\theta_2(t) - \theta_1(t)) + mgL \sin \theta_1(t)$$

$$J_2 \ddot{\theta}_2 = \tau(t) - k(\theta_2(t) - \theta_1(t))$$

# state equation

robotic manipulator

$$J_1 \ddot{\theta}_1(t) = k(\theta_2(t) - \theta_1(t)) + mgL \sin \theta_1(t)$$
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$$\dot{x}(t) = f(x(t), u(t)) \quad x = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \quad u = \tau$$

# state equation

robotic manipulator

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$$\dot{x}(t) = f(x(t), u(t)) \quad x = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \quad u = \tau$$

$$\frac{d}{dt} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ -\frac{k}{J_1} \theta_1 + \frac{k}{J_1} \theta_2 + \frac{mgL}{J_1} \sin \theta_1 \\ \frac{k}{J_2} \theta_1 - \frac{k}{J_2} \theta_2 + \frac{1}{J_2} \tau \end{bmatrix} = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x, u) \end{bmatrix}$$

# equilibrium point

nonlinear dynamical systems and linear approximations

$$\dot{x}(t) = f(x(t), u(t))$$

# equilibrium point

nonlinear dynamical systems and linear approximations

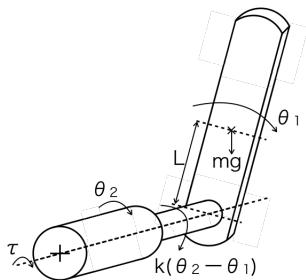
$$\dot{x}(t) = f(x(t), u(t))$$

a pair  $(\bar{x}, \bar{u})$  is an equilibrium point if

$$0 = f(\bar{x}, \bar{u})$$

# equilibrium point

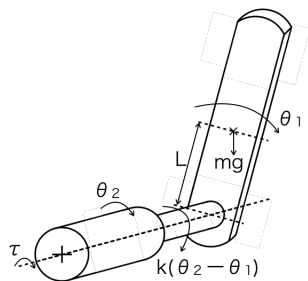
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# equilibrium point

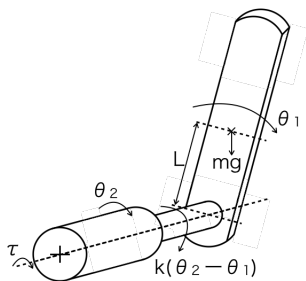
robotic manipulator



$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ -\frac{k}{J_1}\theta_1 + \frac{k}{J_1}\theta_2 + \frac{mgL}{J_1}\sin\theta_1 \\ \frac{k}{J_2}\theta_1 - \frac{k}{J_2}\theta_2 + \frac{1}{J_2}\tau \end{bmatrix}$$

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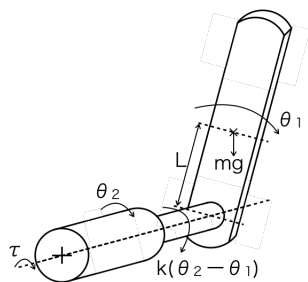
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$$(\bar{x}, \bar{u}) = \left( \begin{bmatrix} \pi \\ \pi \\ 0 \\ 0 \end{bmatrix}, 0 \right)$$



# equilibrium point

robotic manipulator

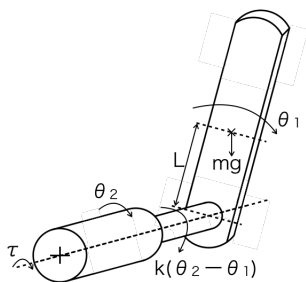


$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ -\frac{k}{J_1}\theta_1 + \frac{k}{J_1}\theta_2 + \frac{mgL}{J_1}\sin\theta_1 \\ \frac{k}{J_2}\theta_1 - \frac{k}{J_2}\theta_2 + \frac{1}{J_2}\tau \end{bmatrix}$$

$$(\bar{x}, \bar{u}) = \left( \begin{bmatrix} \pi \\ \pi \\ 0 \\ 0 \end{bmatrix}, 0 \right) \quad (\bar{x}, \bar{u}) = \left( \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, 0 \right)$$

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$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ -\frac{k}{J_1}\theta_1 + \frac{k}{J_1}\theta_2 + \frac{mgL}{J_1}\sin\theta_1 \\ \frac{k}{J_2}\theta_1 - \frac{k}{J_2}\theta_2 + \frac{1}{J_2}\tau \end{bmatrix}$$

$$(\bar{x}, \bar{u}) = \left( \begin{bmatrix} q - \frac{q}{k} \sin q \\ 0 \\ 0 \end{bmatrix}, -mgL \sin q \right) \quad \text{for all } q \in [\pi, -\pi)$$

# equilibrium point

nonlinear dynamical systems and linear approximations

$$\dot{x}(t) = f(x(t), u(t))$$

a pair  $(\bar{x}, \bar{u})$  is an equilibrium point if

$$0 = f(\bar{x}, \bar{u})$$

# notations

$$h : \mathbb{R} \rightarrow \mathbb{R} \quad y = h(x)$$

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$$y = h(x)$$

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$$= h(x) + \frac{dh}{dx}(x)\delta x + o(\delta x)$$

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$$\frac{\partial h}{\partial x}(x) =$$

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=



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$$h : \mathbb{R}^n \rightarrow \mathbb{R} \quad y = h(x) \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \delta x = \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \vdots \\ \delta x_n \end{bmatrix}$$

$$\begin{aligned} y + \delta y &= h(x + \delta x) \\ &= h(x) + \frac{\partial h}{\partial x_1}(x)\delta x_1 + \frac{\partial h}{\partial x_2}(x)\delta x_2 + \cdots + \frac{\partial h}{\partial x_n}(x)\delta x_n \\ &\quad + o(\delta x_1) + o(\delta x_2) + \cdots o(\delta x_n) \end{aligned}$$

=

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$$= h(x) + \begin{bmatrix} \frac{\partial h}{\partial x_1}(x) & \frac{\partial h}{\partial x_2}(x) & \cdots & \frac{\partial h}{\partial x_n}(x) \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \vdots \\ \delta x_n \end{bmatrix} + o(\delta x)$$

$$= h(x) + \left( \frac{\partial h}{\partial x}(x) \right)^{\text{T}} \delta x + o(\delta x) \quad \delta y \approx \left( \frac{\partial h}{\partial x}(x) \right)^{\text{T}} \delta x$$

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$$\begin{array}{ccc} \frac{\partial h_1}{\partial x_1}(x) & \frac{\partial h_1}{\partial x_2}(x) & \dots & \frac{\partial h_1}{\partial x_n}(x) \\ \frac{\partial h_2}{\partial x_1}(x) & \frac{\partial h_2}{\partial x_2}(x) & \dots & \frac{\partial h_2}{\partial x_n}(x) \\ \vdots & & & \end{array}$$



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$$\frac{\partial h_2}{\partial x_1}(x) \quad \frac{\partial h_2}{\partial x_2}(x) \quad \dots \quad \frac{\partial h_2}{\partial x_n}(x)$$

$$\vdots$$

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$$y + \delta y = h(x + \delta x)$$

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$$\begin{bmatrix} \delta y_1 \\ \delta y_2 \\ \vdots \\ \delta y_m \end{bmatrix} \approx$$

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$$\begin{bmatrix} \delta y_1 \\ \delta y_2 \\ \vdots \\ \delta y_m \end{bmatrix} \approx \begin{bmatrix} \left(\frac{\partial h_1}{\partial x}(x)\right)^T \delta x \\ \left(\frac{\partial h_2}{\partial x}(x)\right)^T \delta x \\ \vdots \\ \left(\frac{\partial h_m}{\partial x}(x)\right)^T \delta x \end{bmatrix} =$$



## notations

$$h : \mathbb{R}^n \rightarrow \mathbb{R}^m \quad y = h(x) \quad \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} h_1(x) \\ h_2(x) \\ \vdots \\ h_m(x) \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \delta x = \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \vdots \\ \delta x_n \end{bmatrix}$$

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=

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## notations

$$h : \mathbb{R} \rightarrow \mathbb{R} \quad y = h(x) \quad \delta y \approx \frac{dh}{dx}(x) \delta x$$

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# linear approximations

## nonlinear dynamical systems and linear approximations

$$\dot{x}(t) = f(x(t), u(t)) \quad f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$$

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$$\delta \dot{x}(t) = \underbrace{f(\bar{x}, \bar{u})}_{=0} + \underbrace{\left( \frac{\partial f}{\partial x}(\bar{x}, \bar{u}) \right)^T}_{=A} \delta x(t) + \underbrace{\left( \frac{\partial f}{\partial u}(\bar{x}, \bar{u}) \right)^T}_{=B} \delta u(t) + o(\delta x(t)) + o(\delta u(t))$$

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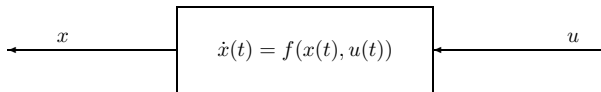
linear approximation around the equilibrium point

$$\delta \dot{x}(t) = A \delta x(t) + B \delta u(t)$$

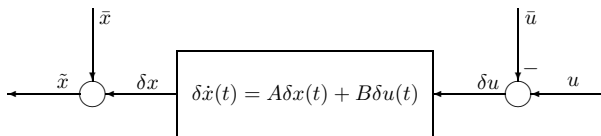
$$A = \left( \frac{\partial f}{\partial x}(\bar{x}, \bar{u}) \right)^T \in \mathbb{R}^{n \times n} \quad B = \left( \frac{\partial f}{\partial u}(\bar{x}, \bar{u}) \right)^T \in \mathbb{R}^{n \times m}$$

# nonlinear dynamical systems and linear approximations

$$\dot{x}(t) = f(x(t), u(t)) \quad x(t) \in \mathbb{R}^n \quad u(t) \in \mathbb{R}^m$$



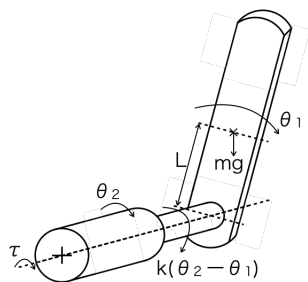
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# equilibrium point

robotic manipulator



$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ -\frac{k}{J_1}\theta_1 + \frac{k}{J_1}\theta_2 + \frac{mgL}{J_1}\sin\theta_1 \\ \frac{k}{J_2}\theta_1 - \frac{k}{J_2}\theta_2 + \frac{1}{J_2}\tau \end{bmatrix}$$

$$(\bar{x}, \bar{u}) = \left( \begin{bmatrix} q - \frac{q}{k} \sin q \\ 0 \\ 0 \end{bmatrix}, -mgL \sin q \right) \quad \text{for all } q \in [\pi, -\pi)$$

# state equation

robotic manipulator

$$J_1 \ddot{\theta}_1(t) = k(\theta_2(t) - \theta_1(t)) + mgL \sin \theta_1(t)$$

$$J_2 \ddot{\theta}_2 = \tau(t) - k(\theta_2(t) - \theta_1(t))$$

$$\dot{x}(t) = f(x(t), u(t)) \quad x = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \quad u = \tau$$

$$\frac{d}{dt} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ -\frac{k}{J_1} \theta_1 + \frac{k}{J_1} \theta_2 + \frac{mgL}{J_1} \sin \theta_1 \\ \frac{k}{J_2} \theta_1 - \frac{k}{J_2} \theta_2 + \frac{1}{J_2} \tau \end{bmatrix} = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x, u) \end{bmatrix}$$

# linear approximation

robotic manipulator

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 0 & -\frac{k}{J_1} + \frac{mgL}{J_1} \cos \theta_1 & \frac{k}{J_2} \\ 0 & 0 & \frac{k}{J_1} & -\frac{k}{J_2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial f}{\partial u} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{J_2} \end{bmatrix}$$

# linear approximation

robotic manipulator

$$(\bar{x}, \bar{u}) = \left( \begin{bmatrix} \bar{q} - \frac{mgL}{k} \sin \bar{q} \\ 0 \\ 0 \end{bmatrix}, -mgL \sin \bar{q} \right) \quad \text{for some } \bar{q} \in [\pi, -\pi)$$

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$$\delta \dot{x}(t) = A \delta x + B \delta u$$



# linear approximation

robotic manipulator

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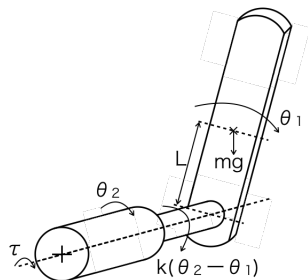
$$\delta \dot{x}(t) = A \delta x + B \delta u$$

$$A = \left( \frac{\partial f}{\partial x}(\bar{x}, \bar{u}) \right)^T = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -\frac{k}{J_1} + \frac{mgL}{J_1} \cos \bar{q} & \frac{k}{J_1} & 0 & 0 & 0 \\ \frac{k}{J_2} & -\frac{k}{J_2} & 0 & 0 & 0 \end{bmatrix}$$

$$B = \left( \frac{\partial f}{\partial u}(\bar{x}, \bar{u}) \right)^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J_2} \end{bmatrix}$$

# numerical simulation

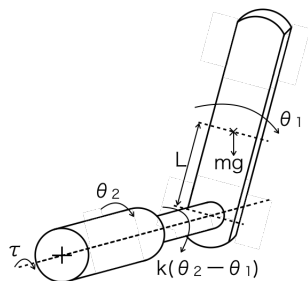
robotic manipulator



$$\frac{d}{dt} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ -\frac{k}{J_1} \theta_1 + \frac{k}{J_1} \theta_2 + \frac{mgL}{J_1} \sin \theta_1 \\ \frac{k}{J_2} \theta_1 - \frac{k}{J_2} \theta_2 + \frac{1}{J_2} \tau \end{bmatrix}$$

# numerical simulation

## robotic manipulator

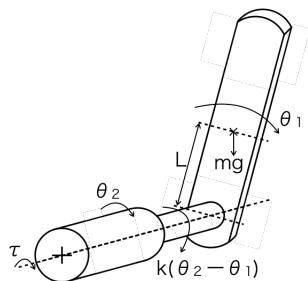


$$\frac{d}{dt} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ -\frac{k}{J_1}\theta_1 + \frac{k}{J_1}\theta_2 + \frac{mgL}{J_1}\sin\theta_1 \\ \frac{k}{J_2}\theta_1 - \frac{k}{J_2}\theta_2 + \frac{1}{J_2}\tau \end{bmatrix}$$

$$J_1 = J_2 = m = k = L = g = 1$$

# numerical simulation

## robotic manipulator



$$\frac{d}{dt} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ -\frac{k}{J_1}\theta_1 + \frac{k}{J_1}\theta_2 + \frac{mgL}{J_1}\sin\theta_1 \\ \frac{k}{J_2}\theta_1 - \frac{k}{J_2}\theta_2 + \frac{1}{J_2}\tau \end{bmatrix}$$

$$J_1 = J_2 = m = k = L = g = 1$$

$$(\bar{x}, \bar{u}) = \left( \begin{bmatrix} \bar{q} \\ \bar{q} - \frac{mgL}{k} \sin \bar{q} \\ 0 \\ 0 \end{bmatrix}, -mgL \sin \bar{q} \right) \quad \text{for some } \bar{q} \in [\pi, -\pi)$$

$$\bar{q} = \pi \text{ [rad]}$$

# numerical simulation

robotic manipulator

$$(\bar{x}, \bar{u}) = \left( \begin{bmatrix} \pi \\ \pi \\ 0 \\ 0 \end{bmatrix}, 0 \right)$$

# numerical simulation

robotic manipulator

$$(\bar{x}, \bar{u}) = \left( \begin{bmatrix} \pi \\ \pi \\ 0 \\ 0 \end{bmatrix}, 0 \right) \quad x(0) = \begin{bmatrix} 11\pi/12 \\ 11\pi/12 \\ 0 \\ 0 \end{bmatrix} \quad (\delta x(0) = \begin{bmatrix} -\pi/12 \\ -\pi/12 \\ 0 \\ 0 \end{bmatrix})$$

# numerical simulation

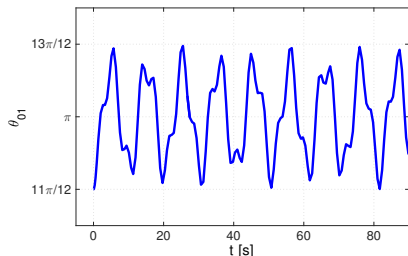
robotic manipulator

$$(\bar{x}, \bar{u}) = \left( \begin{bmatrix} \pi \\ \pi \\ 0 \\ 0 \end{bmatrix}, 0 \right) \quad x(0) = \begin{bmatrix} 11\pi/12 \\ 11\pi/12 \\ 0 \\ 0 \end{bmatrix} \quad (\delta x(0) = \begin{bmatrix} -\pi/12 \\ -\pi/12 \\ 0 \\ 0 \end{bmatrix}) \quad u(\cdot) = 0$$

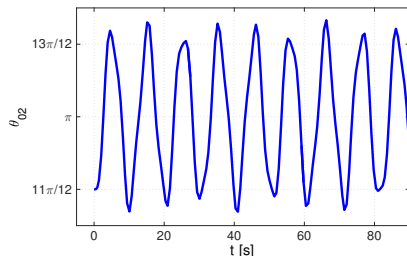
# numerical simulation

robotic manipulator

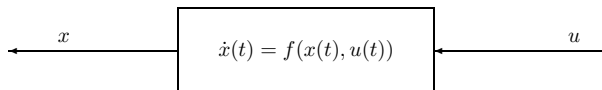
$$(\bar{x}, \bar{u}) = \left( \begin{bmatrix} \pi \\ \pi \\ 0 \\ 0 \end{bmatrix}, 0 \right) \quad x(0) = \begin{bmatrix} 11\pi/12 \\ 11\pi/12 \\ 0 \\ 0 \end{bmatrix} \quad (\delta x(0) = \begin{bmatrix} -\pi/12 \\ -\pi/12 \\ 0 \\ 0 \end{bmatrix}) \quad u(\cdot) = 0$$



$\theta_1$



$\theta_2$



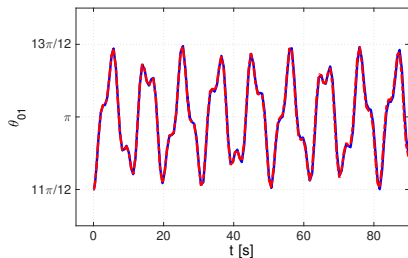


# numerical simulation

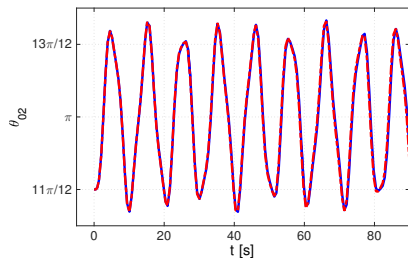
robotic manipulator

$$(\bar{x}, \bar{u}) = \left( \begin{bmatrix} \pi \\ \pi \\ 0 \\ 0 \end{bmatrix}, 0 \right) \quad x(0) = \begin{bmatrix} 11\pi/12 \\ 11\pi/12 \\ 0 \\ 0 \end{bmatrix}$$

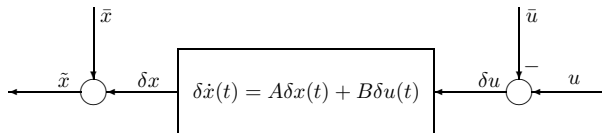
$$(\delta x(0) = \begin{bmatrix} -\pi/12 \\ -\pi/12 \\ 0 \\ 0 \end{bmatrix}) \quad u(\cdot) = 0$$



$\theta_1$



$\theta_2$



# numerical simulation

robotic manipulator

$$(\bar{x}, \bar{u}) = \left( \begin{bmatrix} \pi \\ \pi \\ 0 \\ 0 \end{bmatrix}, 0 \right)$$

# numerical simulation

robotic manipulator

$$(\bar{x}, \bar{u}) = \left( \begin{bmatrix} \pi \\ \pi \\ 0 \\ 0 \end{bmatrix}, 0 \right) \quad x(0) = \begin{bmatrix} \pi/2 \\ \pi/2 \\ 0 \\ 0 \end{bmatrix} \quad (\delta x(0) = \begin{bmatrix} -\pi/2 \\ -\pi/2 \\ 0 \\ 0 \end{bmatrix})$$

# numerical simulation

robotic manipulator

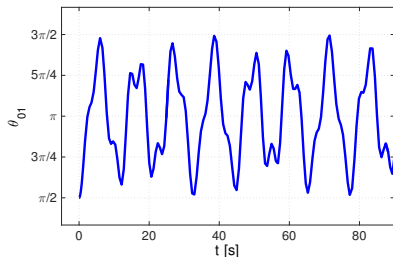
$$(\bar{x}, \bar{u}) = \left( \begin{bmatrix} \pi \\ \pi \\ 0 \\ 0 \end{bmatrix}, 0 \right) \quad x(0) = \begin{bmatrix} \pi/2 \\ \pi/2 \\ 0 \\ 0 \end{bmatrix} \quad (\delta x(0) = \begin{bmatrix} -\pi/2 \\ -\pi/2 \\ 0 \\ 0 \end{bmatrix}) \quad u(\cdot) = 0$$

# numerical simulation

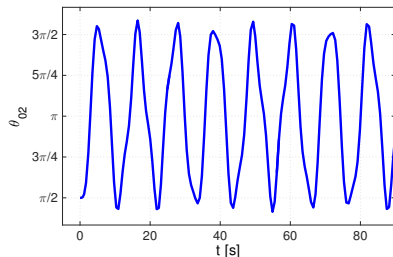
robotic manipulator

$$(\bar{x}, \bar{u}) = \left( \begin{bmatrix} \pi \\ \pi \\ 0 \\ 0 \end{bmatrix}, 0 \right) \quad x(0) = \begin{bmatrix} \pi/2 \\ \pi/2 \\ 0 \\ 0 \end{bmatrix}$$

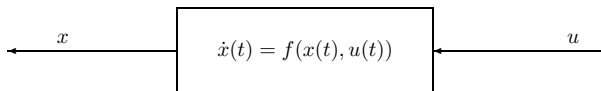
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$\theta_1$



$\theta_2$

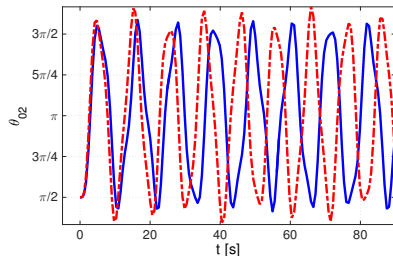
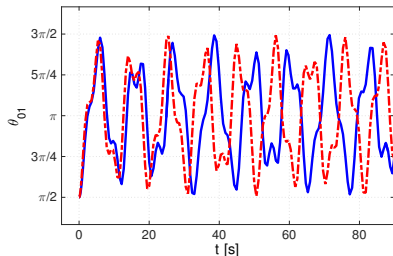


# numerical simulation

robotic manipulator

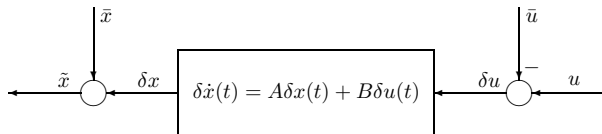
$$(\bar{x}, \bar{u}) = \left( \begin{bmatrix} \pi \\ \pi \\ 0 \\ 0 \end{bmatrix}, 0 \right) \quad x(0) = \begin{bmatrix} \pi/2 \\ \pi/2 \\ 0 \\ 0 \end{bmatrix}$$

$$(\delta x(0) = \begin{bmatrix} -\pi/2 \\ -\pi/2 \\ 0 \\ 0 \end{bmatrix}) \quad u(\cdot) = 0$$



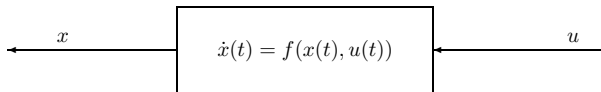
$\theta_1$

$\theta_2$

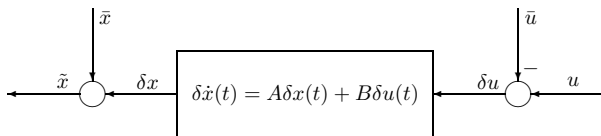


# nonlinear dynamical systems and linear approximations

$$\dot{x}(t) = f(x(t), u(t)) \quad x(t) \in \mathbb{R}^n \quad u(t) \in \mathbb{R}^m$$



$$\delta\dot{x}(t) = A\delta x(t) + B\delta u(t) \quad A \in \mathbb{R}^{n \times n} \quad B \in \mathbb{R}^{n \times m}$$



$$A = \left( \frac{\partial f}{\partial x}(\bar{x}, \bar{u}) \right)^T \quad B = \left( \frac{\partial f}{\partial u}(\bar{x}, \bar{u}) \right)^T$$

# contents

## optimal control

- ▶ nonlinear dynamical systems and linear approximations
- ▶ dynamic programming
- ▶ the principle of optimality
- ▶ optimal control of finite state systems
- ▶ optimal control of discrete-time systems
- ▶ optimal control of continuous-time systems
- ▶ optimal control of linear systems
- ▶ decentralized optimal control
  - ▶ decentralization and integration via mechanism design