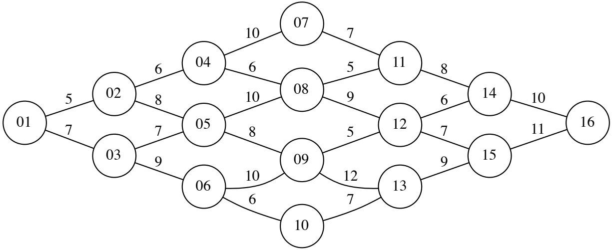
Advanced Control Systems Engineering I:

Optimal Control

finite state systems

optimal control problem



$$X = \{x_1, x_2, \ldots, x_{16}\} \qquad u(t) \in U = \{u_{\mathrm{u}}, u_{\mathrm{d}}\}$$

$$t \in [\,0, 1, 2, 3, 4, 5, 6\,]$$

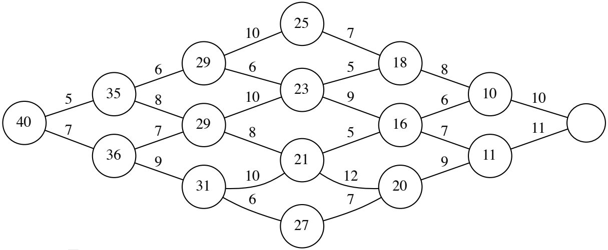
$$\inf_{u(\cdot)} J(0, x_1; u(\cdot)) = \inf_{u(\cdot)} \sum_{\tau=0}^{6} \ell(x(\tau), u(\tau))$$

minimum-cost path problem

- multistage decision process

finite state systems

optimal control problem



define $V : X \to \mathbb{R}$:

$$
\begin{array}{llll}
V(x_1) = 40 & V(x_5) = 29 & V(x_9) = 21 & V(x_{13}) = 20 \\
V(x_2) = 35 & V(x_6) = 31 & V(x_{10}) = 27 & V(x_{14}) = 10 \\
V(x_3) = 36 & V(x_7) = 25 & V(x_{11}) = 18 & V(x_{15}) = 11 \\
V(x_4) = 29 & V(x_8) = 23 & V(x_{12}) = 16 & V(x_{16}) = 0
\end{array}
$$

$V(x_i)$ provides the optimal cost starting from x_i $\qquad$ V: cost-to-go

finite state systems

optimal control problem

$$x(t) \in X = \{x_1, x_2, \dots, x_n\} \qquad u(t) \in U = \{u_1, u_2, \dots, u_m\}$$

$$t \in \{t_0, t_0 + 1, \dots, t_\mathrm{f}\}$$

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

$$\inf_{u(\cdot)} J(t_0, x_0; u(\cdot))$$

finite state systems

optimal control problem

$$x(t) \in X = \{x_1, x_2, \ldots, x_n\} \qquad u(t) \in U = \{u_1, u_2, \ldots, u_m\}$$

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$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

$$\inf_{u(\cdot)} J(t_0, x_0; u(\cdot))$$

$\phi(x_i, u_j)$	u_1	u_2	$\cdots$	u_m
x_1	x_3	x_{n-2}	$\cdots$	x_1
x_2	x_2	x_8	$\cdots$	x_n
$\vdots$				
x_n	x_5	x_{n-7}	$\cdots$	x_2

finite state systems

optimal control problem

$$x(t) \in X = \{x_1, x_2, \ldots, x_n\} \qquad u(t) \in U = \{u_1, u_2, \ldots, u_m\}$$

$$t \in \{t_0, t_0 + 1, \ldots, t_\mathrm{f}\}$$

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

$$\inf_{u(\cdot)} J(t_0, x_0; u(\cdot))$$

$\ell(x_i, u_j)$	u_1	u_2	$\cdots$	u_m
x_1	3	2	$\cdots$	-1
x_2	2	-2	$\cdots$	6
$\vdots$				
x_n	-1	5	$\cdots$	1.2

	$\ell_\mathrm{f}(x_i)$
x_1	3
x_2	2
$\vdots$	
x_n	-1

finite state systems

optimal control problem

$$x(t) \in X = \{x_1, x_2, \ldots, x_n\}$$
$$u(t) \in U = \{u_1, u_2, \ldots, u_m\}$$
$$t \in T = \{t_0, t_0 + 1, \ldots, t_\mathrm{f}\}$$

finite state systems

optimal control problem

$$
\begin{aligned}
x(t) &\in X = \{x_1, x_2, \ldots, x_n\} \\
u(t) &\in U = \{u_1, u_2, \ldots, u_m\} \\
t &\in T = \{t_0, t_0 + 1, \ldots, t_{\mathrm{f}}\}
\end{aligned}
$$

$$
\begin{aligned}
\phi :&\ X \times U \to X & \qquad & x(t+1) = \phi(x(t), u(t)) \\
\ell :&\ X \times U \to \mathbb{R} & & \ell(x(t), u(t)) \\
\ell_{\mathrm{f}} :&\ X \to \mathbb{R} & & \ell_{\mathrm{f}}(x(t_{\mathrm{f}}))
\end{aligned}
$$

finite state systems

optimal control problem

$$
\begin{aligned}
x(t) &\in X = \{x_1, x_2, \ldots, x_n\} \\
u(t) &\in U = \{u_1, u_2, \ldots, u_m\} \\
t &\in T = \{t_0, t_0 + 1, \ldots, t_\mathrm{f}\}
\end{aligned}
$$

$$
\begin{aligned}
\phi &: \; X \times U \to X & \qquad & x(t+1) = \phi(x(t), u(t)) \\
\ell &: \; X \times U \to \mathbb{R} & & \ell(x(t), u(t)) \\
\ell_\mathrm{f} &: \; X \to \mathbb{R} & & \ell_\mathrm{f}(x(t_\mathrm{f}))
\end{aligned}
$$

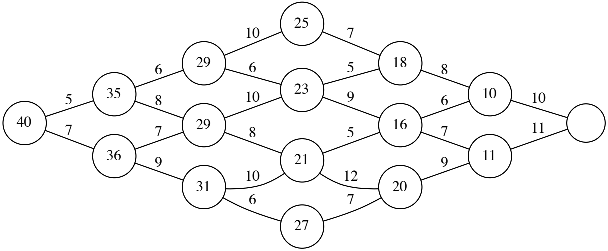
Let $x(t_0) = x_0 \in X$ be given, and consider the optimal control problem:

$$
J(t_0, x_0; u(\cdot)) = \sum_{\tau = t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))
$$

$$
\inf_{\substack{u(\tau) \in U \\ \tau \in T}} J(t_0, x_0; u(\cdot))
$$

finite state systems

optimal control problem



(naive) computational complexities

- # of possible paths: 20 DP had to find only: 15

$n \times n$	4	5	6	7	8
# of paths	20	70	252	724	2632
DP computations	15	24	35	48	63

computational complexity

finite state systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_f-1} \ell(x(\tau), u(\tau)) + \ell_f(x(t_f))$$

brute force search:

$$\begin{aligned} x(t) &\in X = \{x_1, x_2, \ldots, x_n\} \qquad && |X| : \quad \text{cardinality of } X \\ u(t) &\in U = \{u_1, u_2, \ldots, u_m\} && |U| \\ t &\in T = \{t_0, t_0 + 1, \ldots, t_f\} && |T| \end{aligned}$$

computational complexity

finite state systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

brute force search:

$$O(|U|^{|T|} \times |T| \times |X|)$$

$x(t) \in X = \{x_1, x_2, \dots, x_n\}$	$\lvert X\rvert$:	cardinality of X
$u(t) \in U = \{u_1, u_2, \dots, u_m\}$	$\lvert U\rvert$	
$t \in T = \{t_0, t_0+1, \dots, t_\mathrm{f}\}$	$\lvert T\rvert$	

the cost-to-go

finite state systems

Let $x(t_0) = x_0 \in X$ be given, and consider the optimal control problem:

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

$$\inf_{\substack{u(\tau)\in U \\ \tau\in T}} J(t_0, x_0; u(\cdot))$$

the cost-to-go

finite state systems

Let $x(t_0) = x_0 \in X$ be given, and consider the optimal control problem:

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define the cost-to-go:

the cost-to-go

finite state systems

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$$\inf_{\substack{u(\tau)\in U \\ \tau\in T}} J(t_0, x_0; u(\cdot))$$

define the cost-to-go:

$$V : \; T \times X \to \mathbb{R} \qquad\qquad V(t, x) = \inf_{\substack{u(\tau)\in U \\ \tau\in\{t, t+1, \ldots, t_\mathrm{f}\}}} J(t, x; u(\cdot))$$

the cost-to-go

finite state systems

Let $x(t_0) = x_0 \in X$ be given, and consider the optimal control problem:

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

$$\inf_{\substack{u(\tau)\in U \\ \tau\in T}} J(t_0, x_0; u(\cdot))$$

define the cost-to-go:

$$V : \ T \times X \to \mathbb{R} \qquad V(t,x) = \inf_{\substack{u(\tau)\in U \\ \tau\in\{t,t+1,\ldots,t_\mathrm{f}\}}} J(t, x; u(\cdot))$$

computing the cost-to-go $V(t_0, x_0)$ from the initial state x_0 at the initial time t_0 essentially amounts to minimize the cost $J(t_0, x_0; u(\cdot))$

the cost-to-go

finite state systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

$$V:\ T \times X \to \mathbb{R} \qquad\qquad V(t,x) = \inf_{\substack{u(\tau)\in U \\ \tau \in \{t, t+1, \ldots, t_\mathrm{f}\}}} J(t, x; u(\cdot))$$

the cost-to-go

finite state systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_f-1} \ell(x(\tau), u(\tau)) + \ell_f(x(t_f))$$

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example: $\quad |X| = 5 \quad |U| = 3 \quad |T| = 3$

the cost-to-go

finite state systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

$$V: \; T \times X \to \mathbb{R} \qquad V(t,x) = \inf_{\substack{u(\tau)\in U \\ \tau \in \{t, t+1, \ldots, t_\mathrm{f}\}}} J(t, x; u(\cdot))$$

example: $|X| = 5 \quad |U| = 3 \quad |T| = 3$

$\phi(x_i, u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i, u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$\ell_\mathrm{f}(x_i)$
x_1	1
x_2	2
x_3	3
x_4	4
x_5	5

	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1			
x_2			
x_3			
x_4			
x_5			

the cost-to-go

finite state systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_f-1} \ell(x(\tau), u(\tau)) + \ell_f(x(t_f))$$

$$V : \; T \times X \to \mathbb{R} \qquad\qquad V(t,x) = \inf_{\substack{u(\tau)\in U \\ \tau\in\{t,t+1,\dots,t_f\}}} J(t,x;u(\cdot))$$

the cost-to-go

finite state systems

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If $t = t_f$:

the cost-to-go

finite state systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

$$V : \; T \times X \to \mathbb{R} \qquad\qquad V(t, x) = \inf_{\substack{u(\tau) \in U \\ \tau \in \{t, t+1, \dots, t_\mathrm{f}\}}} J(t, x; u(\cdot))$$

If $t = t_\mathrm{f}$:

$$\begin{aligned} V(t_\mathrm{f}, x) &= \inf_{u(t_\mathrm{f}) \in U} J(t_\mathrm{f}, x; u(t_\mathrm{f})) \\ &= \end{aligned}$$

the cost-to-go

finite state systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

$$V : \; T \times X \to \mathbb{R} \qquad\qquad V(t,x) = \inf_{\substack{u(\tau)\in U \\ \tau\in\{t,t+1,\dots,t_\mathrm{f}\}}} J(t,x;u(\cdot))$$

If $t = t_\mathrm{f}$:

$$\begin{aligned} V(t_\mathrm{f}, x) &= \inf_{u(t_\mathrm{f})\in U} J(t_\mathrm{f}, x; u(t_\mathrm{f})) \\ &= \inf_{u(t_\mathrm{f})\in U} \underbrace{\ell_\mathrm{f}(x)}_{\text{independent of } u(t_\mathrm{f})} = \ell_\mathrm{f}(x) \end{aligned}$$

the cost-to-go

finite state systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

$$V : \; T \times X \to \mathbb{R} \qquad\qquad V(t, x) = \inf_{\substack{u(\tau)\in U \\ \tau\in\{t,t+1,\ldots,t_\mathrm{f}\}}} J(t, x; u(\cdot))$$

the cost-to-go

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If $t = t_\mathrm{f} - 1$:

the cost-to-go

finite state systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

$$V: \; T \times X \to \mathbb{R} \qquad\qquad V(t,x) = \inf_{\substack{u(\tau) \in U \\ \tau \in \{t, t+1, \ldots, t_\mathrm{f}\}}} J(t, x; u(\cdot))$$

If $t = t_\mathrm{f} - 1$:

$$\begin{aligned} V(t_\mathrm{f}-1, x) &= \inf_{u(t_\mathrm{f}-1), u(t_\mathrm{f}) \in U} J(t_\mathrm{f}-1, x; u(\cdot)) \\ &= \end{aligned}$$

the cost-to-go

finite state systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

$$V : \; T \times X \to \mathbb{R} \qquad\qquad V(t,x) = \inf_{\substack{u(\tau)\in U \\ \tau \in \{t, t+1, \dots, t_\mathrm{f}\}}} J(t, x; u(\cdot))$$

If $t = t_\mathrm{f} - 1$:

$$\begin{aligned} V(t_\mathrm{f}-1, x) &= \inf_{u(t_\mathrm{f}-1), u(t_\mathrm{f}) \in U} J(t_\mathrm{f}-1, x; u(\cdot)) \\ &= \inf_{u(t_\mathrm{f}-1), u(t_\mathrm{f}) \in U} \{ \underbrace{\ell(x(t_\mathrm{f}-1), u(t_\mathrm{f}-1))}_{\text{independent of } u(t_\mathrm{f})} + \underbrace{\ell_\mathrm{f}(x(t_\mathrm{f}))}_{\substack{\text{depend on both} \\ u(t_\mathrm{f}-1) \text{ and } u(t_\mathrm{f})}} \} \\ &= \end{aligned}$$

the cost-to-go

finite state systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

$$V : \; T \times X \to \mathbb{R} \qquad\qquad V(t, x) = \inf_{\substack{u(\tau)\in U \\ \tau\in\{t,t+1,\ldots,t_\mathrm{f}\}}} J(t, x; u(\cdot))$$

If $t = t_\mathrm{f} - 1$:

$$\begin{aligned}
V(t_\mathrm{f}-1, x) &= \inf_{u(t_\mathrm{f}-1),u(t_\mathrm{f})\in U} J(t_\mathrm{f}-1, x; u(\cdot)) \\
&= \inf_{u(t_\mathrm{f}-1),u(t_\mathrm{f})\in U} \{ \underbrace{\ell(x(t_\mathrm{f}-1), u(t_\mathrm{f}-1))}_{\text{independent of } u(t_\mathrm{f})} + \underbrace{\ell_\mathrm{f}(x(t_\mathrm{f}))}_{\substack{\text{depend on both} \\ u(t_\mathrm{f}-1) \text{ and } u(t_\mathrm{f})}} \} \\
&= \inf_{u\in U} \{ \ell(x, u) + V(t_\mathrm{f}, \phi(x, u)) \}
\end{aligned}$$

the cost-to-go

finite state systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_f-1} \ell(x(\tau), u(\tau)) + \ell_f(x(t_f))$$

$$V : \; T \times X \to \mathbb{R} \qquad\qquad V(t, x) = \inf_{\substack{u(\tau) \in U \\ \tau \in \{t, t+1, \dots, t_f\}}} J(t, x; u(\cdot))$$

the cost-to-go

finite state systems

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$$V : \; T \times X \to \mathbb{R} \qquad\qquad V(t, x) = \inf_{\substack{u(\tau) \in U \\ \tau \in \{t, t+1, \ldots, t_\mathrm{f}\}}} J(t, x; u(\cdot))$$

For $t < t_\mathrm{f}$:

the cost-to-go

finite state systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

$$V : \; T \times X \to \mathbb{R} \qquad V(t, x) = \inf_{\substack{u(\tau) \in U \\ \tau \in \{t, t+1, \ldots, t_\mathrm{f}\}}} J(t, x; u(\cdot))$$

For $t < t_\mathrm{f}$:

$$\begin{aligned} V(t, x) &= \inf_{\substack{u(\tau) \in U \\ \tau \in \{t, t+1, \ldots, t_\mathrm{f}\}}} J(t, x; u(\cdot)) \\ &= \inf_{u \in U} \{\ell(x, u) + V(t+1, \phi(x, u))\} \end{aligned}$$

Bellman equation

finite state systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

$$V : \; T \times X \to \mathbb{R} \qquad\qquad V(t, x) = \inf_{\substack{u(\tau)\in U \\ \tau\in\{t,t+1,\dots,t_\mathrm{f}\}}} J(t, x; u(\cdot))$$

Bellman equation

finite state systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

$$V : \; T \times X \to \mathbb{R} \qquad\qquad V(t,x) = \inf_{\substack{u(\tau)\in U \\ \tau\in\{t,t+1,\ldots,t_\mathrm{f}\}}} J(t, x; u(\cdot))$$

Bellman equation:

$$V(t_\mathrm{f}, x) = \ell_\mathrm{f}(x) \qquad \text{for all } x \in X$$

$$V(t,x) = \inf_{u\in U}\{\ell(x,u) + V(t+1, \phi(x,u))\}$$

$$\text{for all } x \in X \text{ and all } t \in \{t_0, t_{0+1}, \ldots, t_\mathrm{f} - 1\}$$

computational complexity

finite state systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

brute force search:

$$O(|U|^{|T|} \times |T| \times |X|)$$

$$\begin{aligned} x(t) &\in X = \{x_1, x_2, \ldots, x_n\} & |X| &: \quad \text{cardinality of } X \\ u(t) &\in U = \{u_1, u_2, \ldots, u_m\} & |U| & \\ t &\in T = \{t_0, t_0+1, \ldots, t_\mathrm{f}\} & |T| & \end{aligned}$$

computational complexity

finite state systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

brute force search:

$$O(|U|^{|T|} \times |T| \times |X|)$$

DP algorithm:

$$
\begin{aligned}
x(t) &\in X = \{x_1, x_2, \dots, x_n\} & |X| &: \quad \text{cardinality of } X \\
u(t) &\in U = \{u_1, u_2, \dots, u_m\} & |U| & \\
t &\in T = \{t_0, t_0+1, \dots, t_\mathrm{f}\} & |T| &
\end{aligned}
$$

computational complexity

finite state systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

brute force search:

$$O(|U|^{|T|} \times |T| \times |X|)$$

DP algorithm:

$$O(|U| \times |X| \times |T|)$$

$$\begin{aligned} x(t) &\in X = \{x_1, x_2, \ldots, x_n\} \qquad && |X| : \quad \text{cardinality of } X \\ u(t) &\in U = \{u_1, u_2, \ldots, u_m\} && |U| \\ t &\in T = \{t_0, t_0+1, \ldots, t_\mathrm{f}\} && |T| \end{aligned}$$

state feedback implementation

finite state systems

Let $V: \ T \times X \to \mathbb{R}$ be a solution to

$$V(t_\mathrm{f}, x) = \ell_\mathrm{f}(x) \qquad \text{for all } x \in X$$

$$V(t, x) = \inf_{u \in U} \{\ell(x, u) + V(t+1, \phi(x, u))\}$$

$$\text{for all } x \in X \text{ and all } t \in \{t_0, t_{0+1}, \ldots, t_\mathrm{f} - 1\}$$

state feedback implementation

finite state systems

Let $V: \ T \times X \to \mathbb{R}$ be a solution to

$$V(t_{\mathrm{f}}, x) = \ell_{\mathrm{f}}(x) \qquad \text{for all } x \in X$$

$$V(t, x) = \inf_{u \in U} \{\ell(x, u) + V(t+1, \phi(x, u))\}$$

$$\text{for all } x \in X \text{ and all } t \in \{t_0, t_{0+1}, \ldots, t_{\mathrm{f}} - 1\}$$

For a given x at time t, the optimal input $u(t)$ is given as

$$u(t) = \arg\min_{u \in U} \{\ell(x, u) + V(t+1, \phi(x, u))\}$$

state feedback implementation

finite state systems

Let $V: \ T \times X \to \mathbb{R}$ be a solution to

$$V(t_{\mathrm{f}}, x) = \ell_{\mathrm{f}}(x) \qquad \text{for all } x \in X$$

$$V(t, x) = \inf_{u \in U} \{\ell(x, u) + V(t+1, \phi(x, u))\}$$

$$\text{for all } x \in X \text{ and all } t \in \{t_0, t_{0+1}, \ldots, t_{\mathrm{f}} - 1\}$$

For a given x at time t, the optimal input $u(t)$ is given as

$$u(t) = \arg\min_{u \in U} \{\ell(x, u) + V(t+1, \phi(x, u))\}$$

State feedback control:

$$u(t) = u(x(t)) = \arg\min_{u \in U} \{\underbrace{\ell(x(t), u) + V(t+1, \phi(x(t), u))}_{\text{computed using the measured state } x(t)}\}$$

$$x(t+1) = \phi(x(t), u(t)) \qquad x(t_0) = x_0 \in X$$

contents

optimal control

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