Advanced Control Systems Engineering I:

Optimal Control

contents

optimal control

- nonlinear dynamical systems and linear approximations
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- the principle of optimality
- optimal control of finite state systems
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- optimal control of linear systems
- decentralized optimal control
    - decentralization and integration via mechanism design

finite state systems

optimal control problem

$$x(t) \in X = \{x_1, x_2, \ldots, x_n\} \qquad u(t) \in U = \{u_1, u_2, \ldots, u_m\}$$

$$t \in \{t_0, t_0 + 1, \ldots, t_\mathrm{f}\}$$

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau = t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

$$\inf_{u(\cdot)} J(t_0, x_0; u(\cdot))$$

finite state systems

optimal control problem

$$x(t) \in X = \{x_1, x_2, \dots, x_n\} \qquad u(t) \in U = \{u_1, u_2, \dots, u_m\}$$

$$t \in \{t_0, t_0 + 1, \dots, t_f\}$$

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_f - 1} \ell(x(\tau), u(\tau)) + \ell_f(x(t_f))$$

$$\inf_{u(\cdot)} J(t_0, x_0; u(\cdot))$$

$\phi(x_i, u_j)$	u_1	u_2	$\cdots$	u_m
x_1	x_3	x_{n-2}	$\cdots$	x_1
x_2	x_2	x_8	$\cdots$	x_n
$\vdots$				
x_n	x_5	x_{n-7}	$\cdots$	x_2

finite state systems

optimal control problem

$$x(t) \in X = \{x_1, x_2, \ldots, x_n\} \qquad u(t) \in U = \{u_1, u_2, \ldots, u_m\}$$

$$t \in \{t_0, t_0 + 1, \ldots, t_f\}$$

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_f - 1} \ell(x(\tau), u(\tau)) + \ell_f(x(t_f))$$

$$\inf_{u(\cdot)} J(t_0, x_0; u(\cdot))$$

$\ell(x_i, u_j)$	u_1	u_2	$\cdots$	u_m
x_1	3	2	$\cdots$	-1
x_2	2	-2	$\cdots$	6
$\vdots$				
x_n	-1	5	$\cdots$	1.2

	$\ell_f(x_i)$
x_1	3
x_2	2
$\vdots$	
x_n	-1

finite state systems

optimal control problem

$$
\begin{aligned}
x(t) &\in X = \{x_1, x_2, \ldots, x_n\} \\
u(t) &\in U = \{u_1, u_2, \ldots, u_m\} \\
t &\in T = \{t_0, t_0 + 1, \ldots, t_\mathrm{f}\}
\end{aligned}
$$

$$
\begin{aligned}
\phi &: X \times U \to X & \qquad & x(t+1) = \phi(x(t), u(t)) \\
\ell &: X \times U \to \mathbb{R} & & \ell(x(t), u(t)) \\
\ell_\mathrm{f} &: X \to \mathbb{R} & & \ell_\mathrm{f}(x(t_\mathrm{f}))
\end{aligned}
$$

Let $x(t_0) = x_0 \in X$ be given, and consider the optimal control problem:

$$
J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))
$$

$$
\inf_{\substack{u(\tau)\in U \\ \tau\in T}} J(t_0, x_0; u(\cdot))
$$

the cost-to-go

finite state systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

the cost-to-go

finite state systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

$$V : \; T \times X \to \mathbb{R} \qquad\qquad V(t, x) = \inf_{\substack{u(\tau)\in U \\ \tau\in\{t,t+1,\dots,t_\mathrm{f}\}}} J(t, x; u(\cdot))$$

the cost-to-go

finite state systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_f-1} \ell(x(\tau), u(\tau)) + \ell_f(x(t_f))$$

$$V : \; T \times X \to \mathbb{R} \qquad V(t,x) = \inf_{\substack{u(\tau)\in U \\ \tau\in\{t,t+1,\dots,t_f\}}} J(t,x;u(\cdot))$$

Bellman equation:

$$V(t_f, x) = \ell_f(x) \qquad \text{for all } x \in X$$

$$V(t,x) = \inf_{u\in U}\{\ell(x,u) + V(t+1, \phi(x,u))\}$$

$$\text{for all } x \in X \text{ and all } t \in \{t_0, t_{0+1}, \dots, t_f - 1\}$$

state feedback implementation

finite state systems

Let $V : T \times X \to \mathbb{R}$ be a solution to

$$V(t_\mathrm{f}, x) = \ell_\mathrm{f}(x) \qquad \text{for all } x \in X$$

$$V(t, x) = \inf_{u \in U} \{\ell(x, u) + V(t + 1, \phi(x, u))\}$$

$$\text{for all } x \in X \text{ and all } t \in \{t_0, t_{0+1}, \ldots, t_\mathrm{f} - 1\}$$

state feedback implementation

finite state systems

Let $V: \ T \times X \to \mathbb{R}$ be a solution to

$$V(t_{\mathrm{f}}, x) = \ell_{\mathrm{f}}(x) \qquad \text{for all } x \in X$$

$$V(t, x) = \inf_{u \in U} \{\ell(x, u) + V(t+1, \phi(x, u))\}$$

$$\text{for all } x \in X \text{ and all } t \in \{t_0, t_{0+1}, \ldots, t_{\mathrm{f}} - 1\}$$

For a given x at time t, the optimal input $u(t)$ is given as

$$u(t) = \arg\min_{u \in U} \{\ell(x, u) + V(t+1, \phi(x, u))\}$$

state feedback implementation

finite state systems

Let $V: \; T \times X \to \mathbb{R}$ be a solution to

$$V(t_{\mathrm{f}}, x) = \ell_{\mathrm{f}}(x) \qquad \text{for all } x \in X$$

$$V(t, x) = \inf_{u \in U} \{\ell(x, u) + V(t+1, \phi(x, u))\}$$

$$\text{for all } x \in X \text{ and all } t \in \{t_0, t_{0+1}, \ldots, t_{\mathrm{f}} - 1\}$$

For a given x at time t, the optimal input $u(t)$ is given as

$$u(t) = \arg\min_{u \in U} \{\ell(x, u) + V(t+1, \phi(x, u))\}$$

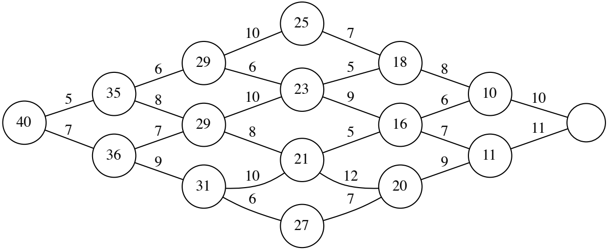
State feedback control:

$$u(t) = u(x(t)) = \arg\min_{u \in U} \{\underbrace{\ell(x(t), u) + V(t+1, \phi(x(t), u))}_{\text{computed using the measured state } x(t)}\}$$

$$x(t+1) = \phi(x(t), u(t)) \qquad x(t_0) = x_0 \in X$$

example00

finite state systems



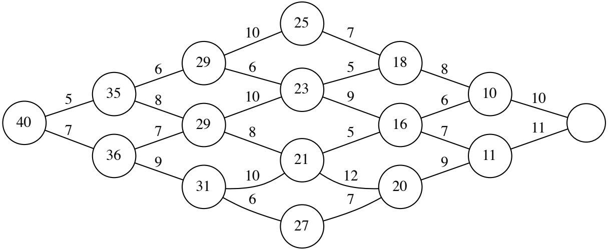
define $V : X \to \mathbb{R}$:

$$
\begin{array}{llll}
V(x_1) = 40 & V(x_5) = 29 & V(x_9) = 21 & V(x_{13}) = 20 \\
V(x_2) = 35 & V(x_6) = 31 & V(x_{10}) = 27 & V(x_{14}) = 10 \\
V(x_3) = 36 & V(x_7) = 25 & V(x_{11}) = 18 & V(x_{15}) = 11 \\
V(x_4) = 29 & V(x_8) = 23 & V(x_{12}) = 16 & V(x_{16}) = 0
\end{array}
$$

$V(x_i)$ provides the optimal cost starting from x_i $\quad V$: cost-to-go

example00

finite state systems



State feedback control:

$$u(t) = u(x(t)) = \arg\min_{u \in U} \{ \underbrace{\ell(x(t), u) + V(t+1, \phi(x(t), u))}_{\text{computed using the measured state } x(t)} \}$$

$$x(t+1) = \phi(x(t), u(t)) \qquad x(t_0) = x_0 \in X$$

example

finite state systems

$$x(t) \in X = \{x_1, x_2, x_4, x_5\} \quad u(t) \in U = \{u_1, u_2, u_3\} \quad t \in T = \{1, 2, 3\}$$

$$\begin{aligned} \phi :&\ X \times U \to X & x(t+1) &= \phi(x(t), u(t)) \\ \ell :&\ X \times U \to \mathbb{R} & &\ell(x(t), u(t)) \\ \ell_{\mathrm{f}} :&\ X \to \mathbb{R} & &\ell_{\mathrm{f}}(x(t_{\mathrm{f}})) \end{aligned}$$

$\phi(x_i, u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i, u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$\ell_{\mathrm{f}}(x_i)$
x_1	1
x_2	2
x_3	3
x_4	4
x_5	5

example

finite state systems

$$x(t) \in X = \{x_1, x_2, x_4, x_5\} \quad u(t) \in U = \{u_1, u_2, u_3\} \quad t \in T = \{1, 2, 3\}$$

Let $x(1) \in X$ be given, and consider the optimal control problem:

$$\begin{aligned} J(1, x(1); u(\cdot)) &= \sum_{\tau=1}^{2} \ell(x(\tau), u(\tau)) + \ell_{\mathrm{f}}(x(3)) \\ &= \ell(x(1), u(1)) + \ell(x(2), u(2)) + \ell_{\mathrm{f}}(x(3)) \end{aligned}$$

$$\inf_{u(1), u(2) \in U} J(1, x(1); u(\cdot))$$

$\phi(x_i, u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i, u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$\ell_{\mathrm{f}}(x_i)$
x_1	1
x_2	2
x_3	3
x_4	4
x_5	5

example

finite state systems

$$x(t) \in X = \{x_1, x_2, x_4, x_5\} \quad u(t) \in U = \{u_1, u_2, u_3\} \quad t \in T = \{1, 2, 3\}$$

Let $x(1) \in X$ be given, and consider the optimal control problem:

$$\begin{aligned} J(1, x(1); u(\cdot)) &= \sum_{\tau=1}^{2} \ell(x(\tau), u(\tau)) + \ell_{\mathrm{f}}(x(3)) \\ &= \ell(x(1), u(1)) + \ell(x(2), u(2)) + \ell_{\mathrm{f}}(x(3)) \end{aligned}$$

$$\inf_{u(1), u(2) \in U} J(1, x(1); u(\cdot))$$

$\phi(x_i, u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i, u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$\ell_{\mathrm{f}}(x_i)$
x_1	1
x_2	2
x_3	3
x_4	4
x_5	5

	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1			
x_2			
x_3			
x_4			
x_5			

Bellman equation

example

$$V(t_{\mathrm{f}}, x) = \ell_{\mathrm{f}}(x) \qquad \text{for all } x \in X$$

$$V(t, x) = \inf_{u \in U} \{\ell(x, u) + V(t + 1, \phi(x, u))\}$$

$$\text{for all } x \in X \text{ and all } t \in \{t_0, t_{0+1}, \ldots, t_{\mathrm{f}} - 1\}$$

Bellman equation

example

$$V(t_f, x) = \ell_f(x) \qquad \text{for all } x \in X$$

$$V(t, x) = \inf_{u \in U} \{\ell(x, u) + V(t + 1, \phi(x, u))\}$$

$$\text{for all } x \in X \text{ and all } t \in \{t_0, t_{0+1}, \ldots, t_f - 1\}$$

$$t = 3: \qquad V(3, x) = \ell_f(x) \qquad \text{for all } x \in X$$

$$t = 2: \qquad V(2, x) = \inf_{u \in U} \{\ell(x, u) + V(3, \phi(x, u))\} \quad \text{for all } x \in X$$

$$t = 1: \qquad V(1, x) = \inf_{u \in U} \{\ell(x, u) + V(2, \phi(x, u))\} \quad \text{for all } x \in X$$

the cost-to-go

example

$$t = 3: \qquad V(3, x) = \ell_{\mathrm{f}}(x) \qquad \text{for all } x \in X$$

	$\ell_{\mathrm{f}}(x_i)$
x_1	1
x_2	2
x_3	3
x_4	4
x_5	5

	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1			
x_2			
x_3			
x_4			
x_5			

the cost-to-go

example

$$t = 3: \qquad V(3, x) = \ell_{\text{f}}(x) \qquad \text{for all } x \in X$$

	$\ell_{\text{f}}(x_i)$
x_1	1
x_2	2
x_3	3
x_4	4
x_5	5

	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1			1
x_2			2
x_3			3
x_4			4
x_5			5

the cost-to-go

example

$$t = 2: \qquad V(2,x) = \inf_{u \in U} \{\ell(x,u) + V(3, \phi(x,u))\} \quad \text{for all } x \in X$$

$\phi(x_i, u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i, u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1			1
x_2			2
x_3			3
x_4			4
x_5			5

the cost-to-go

example

$$t = 2: \qquad V(2, x) = \inf_{u \in U} \{\ell(x, u) + V(3, \phi(x, u))\} \quad \text{for all } x \in X$$

$$V(2, x_1) = \inf_{u \in U} \{\underbrace{\ell(x_1, u)}_{\substack{1\\5\\3}} + \underbrace{V(3, \phi(x_1, u))}_{\substack{3\\1\\5}}\} = 4$$

$\phi(x_i, u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i, u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1			1
x_2			2
x_3			3
x_4			4
x_5			5

the cost-to-go

example

$$t = 2: \qquad V(2,x) = \inf_{u \in U}\{\ell(x,u) + V(3,\phi(x,u))\} \quad \text{for all } x \in X$$

$$V(2,x_2) = \inf_{u \in U}\{\underbrace{\ell(x_2,u)}_{\substack{4\\1\\2}} + \underbrace{V(3,\phi(x_2,u))}_{\substack{4\\3\\2}}\} = 4$$

$\phi(x_i,u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i,u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$V(1,x_i)$	$V(2,x_i)$	$V(3,x_i)$
x_1			1
x_2			2
x_3			3
x_4			4
x_5			5

the cost-to-go

example

$t = 2:$ $$V(2,x) = \inf_{u \in U} \{\ell(x,u) + V(3,\phi(x,u))\} \quad \text{for all } x \in X$$

$$V(2,x_3) = \inf_{u \in U} \{\underbrace{\ell(x_3,u)}_{\substack{2\\3\\1}} + \underbrace{V(3,\phi(x_3,u))}_{\substack{2\\5\\4}}\} = 4$$

$\phi(x_i,u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i,u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$V(1,x_i)$	$V(2,x_i)$	$V(3,x_i)$
x_1			1
x_2			2
x_3			3
x_4			4
x_5			5

the cost-to-go

example

$$t = 2: \qquad V(2, x) = \inf_{u \in U} \{\ell(x, u) + V(3, \phi(x, u))\} \quad \text{for all } x \in X$$

$$V(2, x_4) = \inf_{u \in U} \{\underbrace{\ell(x_4, u)}_{\substack{3\\4\\5}} + \underbrace{V(3, \phi(x_4, u))}_{\substack{1\\2\\1}}\} = 4$$

$\phi(x_i, u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i, u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1			1
x_2			2
x_3			3
x_4			4
x_5			5

the cost-to-go

example

$$t = 2: \qquad V(2,x) = \inf_{u \in U} \{\ell(x,u) + V(3,\phi(x,u))\} \quad \text{for all } x \in X$$

$$V(2,x_5) = \inf_{u \in U} \{\underbrace{\ell(x_5,u)}_{\substack{5\\2\\4}} + \underbrace{V(3,\phi(x_5,u))}_{\substack{5\\4\\3}}\} = 6$$

$\phi(x_i,u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i,u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$V(1,x_i)$	$V(2,x_i)$	$V(3,x_i)$
x_1			1
x_2			2
x_3			3
x_4			4
x_5			5

the cost-to-go

example

$$t = 2: \qquad V(2,x) = \inf_{u\in U}\{\ell(x,u) + V(3,\phi(x,u))\} \quad \text{for all } x \in X$$

$$V(2,x_5) = \inf_{u\in U}\{\underbrace{\ell(x_5,u)}_{\substack{5\\2\\4}} + \underbrace{V(3,\phi(x_5,u))}_{\substack{5\\4\\3}}\} = 6$$

$\phi(x_i,u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i,u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$V(1,x_i)$	$V(2,x_i)$	$V(3,x_i)$
x_1		4	1
x_2		4	2
x_3		4	3
x_4		4	4
x_5		6	5

the cost-to-go

example

$$t = 1 : \qquad V(1, x) = \inf_{u \in U} \{\ell(x, u) + V(2, \phi(x, u))\} \quad \text{for all } x \in X$$

$\phi(x_i, u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i, u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1		4	1
x_2		4	2
x_3		4	3
x_4		4	4
x_5		6	5

the cost-to-go

example

$t = 1:$ $$V(1,x) = \inf_{u \in U} \{\ell(x,u) + V(2,\phi(x,u))\} \quad \text{for all } x \in X$$

$$V(1,x_1) = \inf_{u \in U} \{\underbrace{\ell(x_1,u)}_{\substack{1\\5\\3}} + \underbrace{V(2,\phi(x_1,u))}_{\substack{4\\4\\6}}\} = 5$$

$\phi(x_i,u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i,u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$V(1,x_i)$	$V(2,x_i)$	$V(3,x_i)$
x_1		4	1
x_2		4	2
x_3		4	3
x_4		4	4
x_5		6	5

the cost-to-go

example

$t = 1:$ $$V(1,x) = \inf_{u \in U} \{\ell(x,u) + V(2,\phi(x,u))\} \quad \text{for all } x \in X$$

$$V(1,x_1) = \inf_{u \in U} \{\underbrace{\ell(x_1,u)}_{\substack{1\\5\\3}} + \underbrace{V(2,\phi(x_1,u))}_{\substack{4\\4\\6}}\} = 5$$

$\phi(x_i,u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i,u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$V(1,x_i)$	$V(2,x_i)$	$V(3,x_i)$
x_1	5	4	1
x_2	5	4	2
x_3	5	4	3
x_4	7	4	4
x_5	6	6	5

sample response

example

consider the initial condition $x(1) = x_5$:

$\phi(x_i, u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i, u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$\ell_f(x_i)$
x_1	1
x_2	2
x_3	3
x_4	4
x_5	5

	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1	5	4	1
x_2	5	4	2
x_3	5	4	3
x_4	7	4	4
x_5	6	6	5

sample response

example

consider the initial condition $x(1) = x_5$:

$$x(1) = x_5$$

$\phi(x_i, u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i, u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$\ell_f(x_i)$
x_1	1
x_2	2
x_3	3
x_4	4
x_5	5

	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1	5	4	1
x_2	5	4	2
x_3	5	4	3
x_4	7	4	4
x_5	6	6	5

sample response

example

consider the initial condition $x(1) = x_5$:

$$x(1) = x_5$$

$$u(1) = \arg\min_{u \in U}\{\underbrace{\ell(x_5, u)}_{\substack{5\\2\\4}} + \underbrace{V(2, \phi(x_5, u))}_{\substack{6\\4\\4}}\} \qquad = u_2$$

$\phi(x_i, u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i, u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$\ell_f(x_i)$
x_1	1
x_2	2
x_3	3
x_4	4
x_5	5

	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1	5	4	1
x_2	5	4	2
x_3	5	4	3
x_4	7	4	4
x_5	6	6	5

sample response

example

consider the initial condition $x(1) = x_5$:

$$x(1) = x_5$$

$$u(1) = \arg\min_{u \in U} \{ \underbrace{\ell(x_5, u)}_{\substack{5\\2\\4}} + \underbrace{V(2, \phi(x_5, u))}_{\substack{6\\4\\4}} \} \qquad = u_2$$

$$x(2) = \phi(x_5, u_2) = x_4$$

$\phi(x_i, u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i, u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$\ell_f(x_i)$
x_1	1
x_2	2
x_3	3
x_4	4
x_5	5

	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1	5	4	1
x_2	5	4	2
x_3	5	4	3
x_4	7	4	4
x_5	6	6	5

sample response

example

consider the initial condition $x(1) = x_5$:

$$x(1) = x_5$$

$$u(1) = \arg\min_{u \in U} \{ \underbrace{\ell(x_5, u)}_{\substack{5\\2\\4}} + \underbrace{V(2, \phi(x_5, u))}_{\substack{6\\4\\4}} \} \qquad = u_2$$

$$x(2) = \phi(x_5, u_2) = x_4$$

$$u(2) = \arg\min_{u \in U} \{ \underbrace{\ell(x_4, u)}_{\substack{3\\4\\5}} + \underbrace{V(3, \phi(x_4, u))}_{\substack{1\\2\\1}} \} \qquad = u_1$$

$\phi(x_i, u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i, u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$\ell_f(x_i)$
x_1	1
x_2	2
x_3	3
x_4	4
x_5	5

	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1	5	4	1
x_2	5	4	2
x_3	5	4	3
x_4	7	4	4
x_5	6	6	5

sample response

example

consider the initial condition $x(1) = x_5$:

$$x(1) = x_5$$

$$u(1) = \arg\min_{u \in U} \{ \underbrace{\ell(x_5, u)}_{\substack{5\\2\\4}} + \underbrace{V(2, \phi(x_5, u))}_{\substack{6\\4\\4}} \} = u_2$$

$$x(2) = \phi(x_5, u_2) = x_4$$

$$u(2) = \arg\min_{u \in U} \{ \underbrace{\ell(x_4, u)}_{\substack{3\\4\\5}} + \underbrace{V(3, \phi(x_4, u))}_{\substack{1\\2\\1}} \} = u_1$$

$$x(3) = \phi(x_4, u_1) = x_1$$

$\phi(x_i, u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i, u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$\ell_f(x_i)$
x_1	1
x_2	2
x_3	3
x_4	4
x_5	5

	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1	5	4	1
x_2	5	4	2
x_3	5	4	3
x_4	7	4	4
x_5	6	6	5

sample response

example

consider the initial condition $x(1) = x_5$:

$$\min_{u(1),u(2)\in U} J(1, x_5; u(\cdot)) = \ell(x_5, u_2) + \ell(x_4, u_1) + \ell_f(x_1)$$

$$= 2 + 3 + 1 = 6 = V(1, x_5)$$

$\phi(x_i, u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i, u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$\ell_f(x_i)$
x_1	1
x_2	2
x_3	3
x_4	4
x_5	5

	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1	5	4	1
x_2	5	4	2
x_3	5	4	3
x_4	7	4	4
x_5	6	6	5

computer exercise

finite state systems

computer exercise

finite state systems

Bellman equation:

$$V(t_f, x) = \ell_f(x) \qquad \text{for all } x \in X$$

$$V(t, x) = \inf_{u \in U} \{\ell(x, u) + V(t+1, \phi(x, u))\}$$

$$\text{for all } x \in X \text{ and all } t \in \{t_0, t_{0+1}, \ldots, t_f - 1\}$$

$\phi(x_i, u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i, u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$\ell_f(x_i)$
x_1	1
x_2	2
x_3	3
x_4	4
x_5	5

	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1	5	4	1
x_2	5	4	2
x_3	5	4	3
x_4	7	4	4
x_5	6	6	5

computer exercise

finite state systems

Bellman equation:

$$V(t_\mathrm{f}, x) = \ell_\mathrm{f}(x) \qquad \text{for all } x \in X$$

$$V(t, x) = \inf_{u \in U} \{\ell(x, u) + V(t+1, \phi(x, u))\}$$

$$\text{for all } x \in X \text{ and all } t \in \{t_0, t_{0+1}, \ldots, t_\mathrm{f} - 1\}$$

$\phi(x_i, u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i, u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$\ell_\mathrm{f}(x_i)$
x_1	1
x_2	2
x_3	3
x_4	4
x_5	5

	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1	5	4	1
x_2	5	4	2
x_3	5	4	3
x_4	7	4	4
x_5	6	6	5

data: ϕ, $n \times m$ table (matrix)
ℓ, $n \times m$ table (matrix)
ℓ_f, $n \times 1$ table (vector)

computer exercise

finite state systems

Bellman equation:

$$V(t_\mathrm{f}, x) = \ell_\mathrm{f}(x) \qquad \text{for all } x \in X$$

$$V(t, x) = \inf_{u \in U} \{\ell(x, u) + V(t+1, \phi(x, u))\}$$

$$\text{for all } x \in X \text{ and all } t \in \{t_0, t_{0+1}, \ldots, t_\mathrm{f} - 1\}$$

$\phi(x_i, u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i, u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$\ell_\mathrm{f}(x_i)$
x_1	1
x_2	2
x_3	3
x_4	4
x_5	5

	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1	5	4	1
x_2	5	4	2
x_3	5	4	3
x_4	7	4	4
x_5	6	6	5

data: ϕ, $n \times m$ table (matrix)
ℓ, $n \times m$ table (matrix)
ℓ_f, $n \times 1$ table (vector)

output: V, $n \times m$ table (matirx)

computational complexity

finite state systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_f-1} \ell(x(\tau), u(\tau)) + \ell_f(x(t_f))$$

brute force search:

$$O(|U|^{|T|} \times |T| \times |X|)$$

DP algorithm:

$$O(|U| \times |X| \times |T|)$$

$$x(t) \in X = \{x_1, x_2, \ldots, x_n\} \qquad |X| : \quad \text{cardinality of } X$$

$$u(t) \in U = \{u_1, u_2, \ldots, u_m\} \qquad |U|$$

$$t \in T = \{t_0, t_0 + 1, \ldots, t_f\} \qquad |T|$$

contents

optimal control

- nonlinear dynamical systems and linear approximations
- dynamic programming
- the principle of optimality
- optimal control of finite state systems
- optimal control of discrete-time systems
- optimal control of continuous-time systems
- optimal control of linear systems
- decentralized optimal control
  - decentralization and integration via mechanism design