# Advanced Control Systems Engineering I:

## Optimal Control

# contents

## optimal control systems

- nonlinear dynamical systems and linear approximations
- dynamic programming
- the principle of optimality
- optimal control of finite state systems
- optimal control of discrete-time systems
- optimal control of continuous-time systems
- optimal control of linear systems
- decentralized optimal control
  - decentralization and integration via mechanism design

# finite state systems

## optimal control problem

$$\begin{aligned} x(t) &\in X = \{x_1, x_2, \ldots, x_n\} \\ u(t) &\in U = \{u_1, u_2, \ldots, u_m\} \\ t &\in T = \{t_0, t_0 + 1, \ldots, t_\mathrm{f}\} \end{aligned}$$

# finite state systems

## optimal control problem

$$x(t) \in X = \{x_1, x_2, \dots, x_n\}$$
$$u(t) \in U = \{u_1, u_2, \dots, u_m\}$$
$$t \in T = \{t_0, t_0 + 1, \dots, t_\mathrm{f}\}$$

$$\phi : \ X \times U \to X \qquad x(t+1) = \phi(x(t), u(t))$$
$$\ell : \ X \times U \to \mathbb{R} \qquad \ell(x(t), u(t))$$
$$\ell_\mathrm{f} : \ X \to \mathbb{R} \qquad \ell_\mathrm{f}(x(t_\mathrm{f}))$$

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$$\ell_\mathrm{f} : X \to \mathbb{R} \qquad \ell_\mathrm{f}(x(t_\mathrm{f}))$$

Let $x(t_0) = x_0 \in X$ be given, and consider the optimal control problem:

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

$$\inf_{\substack{u(\tau)\in U \\ \tau \in T}} J(t_0, x_0; u(\cdot))$$

# state feedback implementation

## finite state systems

Let $V: \ T \times X \to \mathbb{R}$ be a solution to

$$V(t_{\mathrm{f}}, x) = \ell_{\mathrm{f}}(x) \qquad \text{for all } x \in X$$

$$V(t, x) = \inf_{u \in U} \{\ell(x, u) + V(t+1, \phi(x, u))\}$$

$$\text{for all } x \in X \text{ and all } t \in \{t_0, t_{0+1}, \ldots, t_{\mathrm{f}} - 1\}$$

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For a given $x$ at time $t$, the optimal input $u(t)$ is given as

$$u(t) = \arg\min_{u \in U} \{\ell(x, u) + V(t+1, \phi(x, u))\}$$

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State feedback control:

$$u(t) = u(x(t)) = \arg\min_{u \in U} \{\underbrace{\ell(x(t), u) + V(t+1, \phi(x(t), u))}_{\text{computed using the measured state } x(t)}\}$$

$$x(t+1) = \phi(x(t), u(t)) \qquad x(t_0) = x_0 \in X$$

# finite state systems

## optimal control problem

$$\begin{aligned} x(t) &\in X = \{x_1, x_2, \ldots, x_n\} \\ u(t) &\in U = \{u_1, u_2, \ldots, u_m\} \\ t &\in T = \{t_0, t_0+1, \ldots, t_\mathrm{f}\} \end{aligned}$$

$$\begin{aligned} \phi &: X \times U \to X & \qquad & x(t+1) = \phi(x(t), u(t)) \\ \ell &: X \times U \to \mathbb{R} & & \ell(x(t), u(t)) \\ \ell_\mathrm{f} &: X \to \mathbb{R} & & \ell_\mathrm{f}(x(t_\mathrm{f})) \end{aligned}$$

Let $x(t_0) = x_0 \in X$ be given, and consider the optimal control problem:

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

$$\inf_{\substack{u(\tau)\in U \\ \tau \in T}} J(t_0, x_0; u(\cdot))$$

# discrete-time systems

## optimal control problem

$$x(t+1) = f(x(t), u(t)) \qquad x(t_0) = x_0 \qquad t \in T = \{t_0, t_0 + 1, \ldots, t_f\}$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

# discrete-time systems

## optimal control problem

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For a given $x(t_0) = x_0 \in \mathbb{R}^n$

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# discrete-time systems

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$$\inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in T}} J(t_0, x_0; u(\cdot))$$

# the cost-to-go

## discrete-time systems

Let $x(t_0) = x_0 \in X$ be given, and consider the optimal control problem:

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

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$$\inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in T}} J(t_0, x_0; u(\cdot))$$

define the cost-to-go:

$$V : \; T \times \mathbb{R}^n \to \mathbb{R} \qquad\qquad V(t,x) = \inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in\{t,t+1,\ldots,t_\mathrm{f}\}}} J(t, x; u(\cdot))$$

# the cost-to-go

## discrete-time systems

Let $x(t_0) = x_0 \in X$ be given, and consider the optimal control problem:

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

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computing the cost-to-go $V(t_0, x_0)$ from the initial state $x_0$ at the initial time $t_0$ essentially amounts to minimize the cost $J(t_0, x_0; u(\cdot))$

# the cost-to-go

## discrete-time systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\text{f}-1} \ell(x(\tau), u(\tau)) + \ell_\text{f}(x(t_\text{f}))$$

$$V:\ T \times \mathbb{R}^n \to \mathbb{R} \qquad V(t,x) = \inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in\{t,t+1,\dots,t_\text{f}\}}} J(t,x;u(\cdot))$$

If $t = t_\text{f}$:

# the cost-to-go

## discrete-time systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

$$V: \; T \times \mathbb{R}^n \to \mathbb{R} \qquad V(t,x) = \inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in\{t,t+1,\dots,t_\mathrm{f}\}}} J(t,x;u(\cdot))$$

If $t = t_\mathrm{f}$:

$$\begin{aligned} V(t_\mathrm{f}, x) &= \inf_{u(t_\mathrm{f})\in\mathbb{R}^m} J(t_\mathrm{f}, x; u(t_\mathrm{f})) \\ &= \inf_{u(t_\mathrm{f})\in\mathbb{R}^m} \underbrace{\ell_\mathrm{f}(x)}_{\text{independent of } u(t_\mathrm{f})} = \ell_\mathrm{f}(x) \end{aligned}$$

# the cost-to-go

## discrete-time systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

$$V : \; T \times \mathbb{R}^n \to \mathbb{R} \qquad\qquad V(t, x) = \inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in\{t,t+1,\ldots,t_\mathrm{f}\}}} J(t, x; u(\cdot))$$

If $t = t_\mathrm{f} - 1$:

# the cost-to-go

## discrete-time systems

If $t = t_f - 1$:

$$V(t_f - 1, x) = \inf_{u(t_f-1), u(t_f) \in \mathbb{R}^m} J(t_f - 1, x; u(\cdot))$$

$$= \inf_{u(t_f-1), u(t_f) \in \mathbb{R}^m} \{ \underbrace{\ell(x(t_f - 1), u(t_f - 1))}_{\text{independent of } u(t_f)} + \underbrace{\ell_f(x(t_f))}_{\substack{\text{depend on both} \\ u(t_f - 1) \text{ and } u(t_f)}} \}$$

$$= \inf_{u(t_f-1) \in \mathbb{R}^m} \{ \ell(x, u(t_f - 1)) + \underbrace{\inf_{u(t_f) \in \mathbb{R}^m} \ell_f(x(t_f))}_{= V(t_f, x(t_f)) = V(t_f, f(x, u(t_f - 1)))} \}$$

$$= \inf_{u(t_f-1) \in \mathbb{R}^m} \{ \ell(x, u(t_f - 1)) + V(t_f, f(x, u(t_f - 1))) \}$$

$$= \inf_{u \in \mathbb{R}^m} \{ \ell(x, u) + V(t_f, f(x, u)) \}$$

# the cost-to-go

## discrete-time systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

$$V : \; T \times \mathbb{R}^n \to \mathbb{R} \qquad\qquad V(t, x) = \inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in\{t,t+1,\ldots,t_\mathrm{f}\}}} J(t, x; u(\cdot))$$

For $t < t_\mathrm{f}$:

# the cost-to-go

## discrete-time systems

For $t < t_\mathrm{f}$:

$$V(t,x) = \inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in\{t,t+1,\ldots,t_\mathrm{f}\}}} J(t,x;u(\cdot))$$

$$= \inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in\{t,t+1,\ldots,t_\mathrm{f}\}}} \{\sum_{\tau=t}^{t_\mathrm{f}-1} \ell(x(\tau),u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))\}$$

$$= \inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in\{t,t+1,\ldots,t_\mathrm{f}\}}} \{ \underbrace{\ell(x,u(t))}_{\substack{\text{independent of } u(\tau), \\ \tau\in\{t+1,t+2,\ldots,t_\mathrm{f}\}}} + \underbrace{\sum_{\tau=t+1}^{t_\mathrm{f}-1} \ell(x(\tau),u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))}_{\text{depend on all } u(\tau),\ \tau\in\{t,t+1,\ldots,t_\mathrm{f}}$$

$$= \inf_{u(t)\in\mathbb{R}^m} \{\ell(x,u(t)) + \underbrace{\inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in\{t+1,t+2,\ldots,t_\mathrm{f}\}}} \{\sum_{\tau=t+1}^{t_\mathrm{f}-1} \ell(x(\tau),u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))}_{= V(t+1,x(t+1)) = V(t+1,f(x,u(t)))}$$

$$= \inf \{\ell(x,u(t)) + V(t+1,f(x,u(t)))\}$$

# the cost-to-go

## discrete-time systems

$$
\begin{aligned}
V(t,x) &= \inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in\{t,t+1,\dots,t_\mathrm{f}\}}} J(t,x;u(\cdot)) \\
&= \inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in\{t,t+1,\dots,t_\mathrm{f}\}}} \Big\{\sum_{\tau=t}^{t_\mathrm{f}-1} \ell(x(\tau),u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))\Big\} \\
&= \inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in\{t,t+1,\dots,t_\mathrm{f}\}}} \Big\{ \underbrace{\ell(x,u(t))}_{\substack{\text{independent of } u(\tau), \\ \tau\in\{t+1,t+2,\dots,t_\mathrm{f}\}}} + \underbrace{\sum_{\tau=t+1}^{t_\mathrm{f}-1} \ell(x(\tau),u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))}_{\text{depend on all } u(\tau),\ \tau\in\{t,t+1,\dots,t_\mathrm{f}\}} \Big\} \\
&= \inf_{u(t)\in\mathbb{R}^m}\Big\{\ell(x,u(t)) + \underbrace{\inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in\{t+1,t+2,\dots,t_\mathrm{f}\}}} \Big\{\sum_{\tau=t+1}^{t_\mathrm{f}-1} \ell(x(\tau),u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))\Big\}}_{=\,V(t+1,x(t+1))\,=\,V(t+1,f(x,u(t)))}\Big\} \\
&= \inf_{u(t)\in\mathbb{R}^m}\{\ell(x,u(t)) + V(t+1,f(x,u(t)))\} \\
&= \inf_{u\in\mathbb{R}^m}\{\ell(x,u) + V(t+1,f(x,u))\}
\end{aligned}
$$

# Bellman equation:
## discrete-time systems

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$$V: \; T \times \mathbb{R}^n \to \mathbb{R} \qquad\qquad V(t,x) = \inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in\{t,t+1,\dots,t_f\}}} J(t,x;u(\cdot))$$

Bellman equation:

$$V(t_f, x) = \ell_f(x) \qquad \text{for all } x \in \mathbb{R}^n$$

$$V(t,x) = \inf_{u\in\mathbb{R}^m} \{\ell(x,u) + V(t+1, f(x,u))\}$$

for all $x \in \mathbb{R}^n$ and all $t \in \{t_0, t_{0+1}, \dots, t_f - 1\}$

# state feedback implementation

## discrete-time systems

Let $V:\ T \times \mathbb{R}^n \to \mathbb{R}$ be a solution to

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