Advanced Control Systems Engineering I:

Optimal Control

contents

- nonlinear dynamical systems and linear approximations
- dynamic programming
- the principle of optimality
- optimal control of finite state systems
- optimal control of discrete-time systems
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- decentralized optimal control
  - decentralization and integration via mechanism design

continuous-time systems

optimal control problem

$$\dot{x}(t) = f(x(t), u(t)) \qquad x(t_0) = x_0 \qquad t \in [\, t_0, t_\mathrm{f} \,]$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_\mathrm{f}} \ell(x(\tau), u(\tau))\, d\tau + \ell_\mathrm{f}(x_\mathrm{f}(t_\mathrm{f}))$$

$$\inf_{u(\cdot)} J(t_0, x_0; u(\cdot))$$

a solution:

$$0 = \frac{\partial V}{\partial t}(t, x) + \inf_{u \in \mathbb{R}^m} \left\{ \ell(x, u) + \left(\frac{\partial V}{\partial x}(t, x) \right)^\mathrm{T} f(x, u) \right\}$$

$$V(t_\mathrm{f}, x) = \ell_\mathrm{f}(x) \qquad \text{for all } x \in \mathbb{R}^n$$

$$u(t) = u(x(t)) = \arg\min_{u \in \mathbb{R}^m} \left\{ \ell(x(t), u) + \left(\frac{\partial V}{\partial x}(t, x(t)) \right)^\mathrm{T} f(x(t), u) \right\}$$

continuous-time systems

optimal control problem

$$\dot{x}(t) = f(x(t), u(t)) \qquad x(t_0) = x_0 \qquad t \in [\, t_0, t_\text{f} \,]$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

continuous-time systems

optimal control problem

$$\dot{x}(t) = f(x(t), u(t)) \qquad x(t_0) = x_0 \qquad t \in [\, t_0, t_\mathrm{f} \,]$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

for a given $x(t_0) = x_0 \in \mathbb{R}^n$

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_\mathrm{f}} \ell(x(\tau), u(\tau))\, d\tau + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

continuous-time systems

optimal control problem

$$\dot{x}(t) = f(x(t), u(t)) \qquad x(t_0) = x_0 \qquad t \in [\, t_0, t_\mathrm{f} \,]$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

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optimal control problem

$$\inf_{\substack{u(\tau) \\ \tau \in [\, t_0, t_\mathrm{f} \,]}} \quad J(t_0, x_0; u(\cdot))$$

the cost-to-go

continuous-time systems

for a given $x(t_0) = x_0 \in \mathbb{R}^n$

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_\mathrm{f}} \ell(x(\tau), u(\tau))\, d\tau + \ell_\mathrm{f}(x_\mathrm{f}(t_\mathrm{f}))$$

optimal control problem

$$\inf_{\substack{u(\tau) \\ \tau \in [\, t_0, t_\mathrm{f} \,]}} \quad J(t_0, x_0; u(\cdot))$$

the cost-to-go

continuous-time systems

for a given $x(t_0) = x_0 \in \mathbb{R}^n$

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_\mathrm{f}} \ell(x(\tau), u(\tau))\, d\tau + \ell_\mathrm{f}(x_\mathrm{f}(t_\mathrm{f}))$$

optimal control problem

$$\inf_{\substack{u(\tau) \\ \tau \in [\, t_0, t_\mathrm{f}\,]}} \quad J(t_0, x_0; u(\cdot))$$

define the cost-to-go

the cost-to-go

continuous-time systems

for a given $x(t_0) = x_0 \in \mathbb{R}^n$

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_\mathrm{f}} \ell(x(\tau), u(\tau))\, d\tau + \ell_\mathrm{f}(x_\mathrm{f}(t_\mathrm{f}))$$

optimal control problem

$$\inf_{\substack{u(\tau) \\ \tau \in [\, t_0, t_\mathrm{f}\,]}} \quad J(t_0, x_0; u(\cdot))$$

define the cost-to-go

$$V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \qquad\qquad V(t, x) = \inf_{\substack{u(\tau) \\ \tau \in [\, t, t_\mathrm{f}\,]}} \quad J(t, x; u(\cdot))$$

computing the cost-to-go $V(t_0, x_0)$ from the initial state x_0 at the initial time t_0 essentially amounts to minimize the cost $J(t_0, x_0; u(\cdot))$.

the cost-to-go

continuous-time systems

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_\mathrm{f}} \ell(x(\tau), u(\tau))\, d\tau + \ell_\mathrm{f}(x_\mathrm{f}(t_\mathrm{f}))$$

$$V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \qquad\qquad V(t, x) = \inf_{\substack{u(\tau) \\ \tau \in [\, t, t_\mathrm{f} \,]}} J(t, x; u(\cdot))$$

the cost-to-go

continuous-time systems

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_\mathrm{f}} \ell(x(\tau), u(\tau))\, d\tau + \ell_\mathrm{f}(x_\mathrm{f}(t_\mathrm{f}))$$

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if $t = t_\mathrm{f}$:

the cost-to-go

continuous-time systems

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_\mathrm{f}} \ell(x(\tau), u(\tau))\, d\tau + \ell_\mathrm{f}(x_\mathrm{f}(t_\mathrm{f}))$$

$$V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \qquad V(t, x) = \inf_{\substack{u(\tau) \\ \tau \in [\, t, t_\mathrm{f}\,]}} J(t, x; u(\cdot))$$

if $t = t_\mathrm{f}$:

$$V(t_\mathrm{f}, x) = \inf_{u(t_\mathrm{f})} J(t_\mathrm{f}, x; u(t_\mathrm{f}))$$

$$=$$

the cost-to-go

continuous-time systems

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_\mathrm{f}} \ell(x(\tau), u(\tau))\, d\tau + \ell_\mathrm{f}(x_\mathrm{f}(t_\mathrm{f}))$$

$$V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \qquad V(t, x) = \inf_{\substack{u(\tau) \\ \tau \in [\, t, t_\mathrm{f}\,]}} J(t, x; u(\cdot))$$

if $t = t_\mathrm{f}$:

$$\begin{aligned} V(t_\mathrm{f}, x) &= \inf_{u(t_\mathrm{f})} J(t_\mathrm{f}, x; u(t_\mathrm{f})) \\ &= \inf_{u(t_\mathrm{f})} \underbrace{\ell_\mathrm{f}(x)}_{\substack{\text{independent} \\ \text{of } u(t_\mathrm{f})}} = \ell_\mathrm{f}(x) \end{aligned}$$

the cost-to-go

continuous-time systems

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_\mathrm{f}} \ell(x(\tau), u(\tau))\, d\tau + \ell_\mathrm{f}(x_\mathrm{f}(t_\mathrm{f}))$$

$$V : \ \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \qquad\qquad V(t, x) = \inf_{\substack{u(\tau) \\ \tau \in [\, t, t_\mathrm{f}\,]}} J(t, x; u(\cdot))$$

the cost-to-go

continuous-time systems

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_\mathrm{f}} \ell(x(\tau), u(\tau))\, d\tau + \ell_\mathrm{f}(x_\mathrm{f}(t_\mathrm{f}))$$

$$V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \qquad\qquad V(t, x) = \inf_{\substack{u(\tau) \\ \tau \in [\, t, t_\mathrm{f}\,]}} J(t, x; u(\cdot))$$

For $t < t_\mathrm{f}$,

the cost-to-go

continuous-time systems

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_\mathrm{f}} \ell(x(\tau), u(\tau))\, d\tau + \ell_\mathrm{f}(x_\mathrm{f}(t_\mathrm{f}))$$

$$V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \qquad V(t, x) = \inf_{\substack{u(\tau) \\ \tau \in [\, t, t_\mathrm{f}\,]}} J(t, x; u(\cdot))$$

For $t < t_\mathrm{f}$, let us pick some small positive constant δt so that $t + \delta t$ is still smaller than t_f:

the cost-to-go

continuous-time systems

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_\mathrm{f}} \ell(x(\tau), u(\tau))\, d\tau + \ell_\mathrm{f}(x_\mathrm{f}(t_\mathrm{f}))$$

$$V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \qquad\qquad V(t, x) = \inf_{\substack{u(\tau) \\ \tau \in [\, t, t_\mathrm{f}\,]}} J(t, x; u(\cdot))$$

For $t < t_\mathrm{f}$, let us pick some small positive constant δt so that $t + \delta t$ is still smaller than t_f:

$$\begin{aligned} V(t, x) &= \inf_{\substack{u(\tau) \\ \tau \in [\, t, t_\mathrm{f}\,]}} J(t, x; u(\cdot)) \\ &= \inf_{\substack{u(\tau) \\ \tau \in [\, t, t+\delta t\,]}} \left\{ \int_t^{t+\delta t} \ell(x(\tau), u(\tau))\, d\tau + V(t + \delta t, x(t + \delta t)) \right\} \end{aligned}$$

This represents the principle of optimality

the cost-to-go

continuous-time systems

$$V(t,x) = \inf_{\substack{u(\tau) \\ \tau \in [\,t, t_\mathrm{f}\,]}} J(t,x;u(\cdot))$$

$$=$$

$$=$$

the cost-to-go

continuous-time systems

$$V(t,x) = \inf_{\substack{u(\tau) \\ \tau\in[\,t,t_{\mathrm{f}}\,]}} \left\{ \int_t^{t_{\mathrm{f}}} \ell(x(\tau),u(\tau))\,d\tau + \ell_{\mathrm{f}}(x(t_{\mathrm{f}})) \right\}$$

$$=$$

$$=$$

the cost-to-go

continuous-time systems

$$V(t,x) = \inf_{\substack{u(\tau) \\ \tau\in[\,t, t_\mathrm{f}\,]}} \left\{ \int_t^{t_\mathrm{f}} \ell(x(\tau),u(\tau))\,d\tau + \ell_\mathrm{f}(x(t_\mathrm{f})) \right\}$$

$$= \inf_{\substack{u(\tau) \\ \tau\in[\,t, t_\mathrm{f}\,]}} \left\{ \underbrace{\int_t^{t+\delta t} \ell(x(\tau),u(\tau))\,d\tau}_{\text{independent of } u(\tau),\ \tau\in[\,t+\delta t,\, t_\mathrm{f}\,]} + \underbrace{\int_{t+\delta t}^{t_\mathrm{f}} \ell(x(\tau),u(\tau))\,d\tau + \ell_\mathrm{f}(x(t_\mathrm{f}))}_{\text{depend on all } u(\tau),\ \tau\in[\,t,\, t_\mathrm{f}\,]} \right\}$$

$$=$$

the cost-to-go

continuous-time systems

$$V(t,x) = \inf_{\substack{u(\tau)\\ \tau\in[\,t,t_f\,]}} \{\int_t^{t_f} \ell(x(\tau),u(\tau))\,d\tau + \ell_f(x(t_f))\}$$

$$= \inf_{\substack{u(\tau)\\ \tau\in[\,t,t_f\,]}} \{\underbrace{\int_t^{t+\delta t} \ell(x(\tau),u(\tau))\,d\tau}_{\substack{\text{independent of } u(\tau),\\ \tau\in[\,t+\delta t,\,t_f\,]}} + \underbrace{\int_{t+\delta t}^{t_f} \ell(x(\tau),u(\tau))\,d\tau + \ell_f(x(t_f))}_{\text{depend on all } u(\tau),\ \tau\in[\,t,\,t_f\,]}\}$$

$$= \inf_{\substack{u(\tau)\\ \tau\in[\,t,t+\delta t\,]}} \{\int_t^{t+\delta t} \ell(x(\tau),u(\tau))\,d\tau$$

$$+ \underbrace{\inf_{\substack{u(\tau)\\ \tau\in[\,t+\delta t,t_f\,]}} \{\int_{t+\delta t}^{t_f} \ell(x(\tau),u(\tau))\,d\tau + \ell_f(x(t_f))\}\}}_{=\,V(t+\delta t,\,x(t+\delta t))}$$

the cost-to-go

continuous-time systems

$$
\begin{aligned}
V(t,x) &= \inf_{\substack{u(\tau)\\ \tau\in[\,t,t_\mathrm{f}\,]}} \Big\{\int_t^{t_\mathrm{f}} \ell(x(\tau),u(\tau))\,d\tau + \ell_\mathrm{f}(x(t_\mathrm{f}))\Big\} \\
&= \inf_{\substack{u(\tau)\\ \tau\in[\,t,t_\mathrm{f}\,]}} \Big\{\underbrace{\int_t^{t+\delta t} \ell(x(\tau),u(\tau))\,d\tau}_{\substack{\text{independent of } u(\tau),\\ \tau\in[\,t+\delta t,\,t_\mathrm{f}\,]}} + \underbrace{\int_{t+\delta t}^{t_\mathrm{f}} \ell(x(\tau),u(\tau))\,d\tau + \ell_\mathrm{f}(x(t_\mathrm{f}))}_{\text{depend on all } u(\tau),\ \tau\in[\,t,\,t_\mathrm{f}\,]}\Big\} \\
&= \inf_{\substack{u(\tau)\\ \tau\in[\,t,t+\delta t\,]}} \Big\{\int_t^{t+\delta t} \ell(x(\tau),u(\tau))\,d\tau + V(t+\delta t, x(t+\delta t))\Big\}
\end{aligned}
$$

the cost-to-go

continuous-time systems

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_\mathrm{f}} \ell(x(\tau), u(\tau))\, d\tau + \ell_\mathrm{f}(x_\mathrm{f}(t_\mathrm{f}))$$

$$V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \qquad V(t, x) = \inf_{\substack{u(\tau) \\ \tau \in [\, t, t_\mathrm{f}\,]}} J(t, x; u(\cdot))$$

For $t < t_\mathrm{f}$, let us pick some small positive constant δt so that $t + \delta t$ is still smaller than t_f:

$$\begin{aligned} V(t, x) &= \inf_{\substack{u(\tau) \\ \tau \in [\, t, t_\mathrm{f}\,]}} J(t, x; u(\cdot)) \\ &= \inf_{\substack{u(\tau) \\ \tau \in [\, t, t+\delta t\,]}} \left\{ \int_t^{t+\delta t} \ell(x(\tau), u(\tau))\, d\tau + V(t + \delta t, x(t + \delta t)) \right\} \end{aligned}$$

This represents the principle of optimality

Bellman equation

finite state systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_f-1} \ell(x(\tau), u(\tau)) + \ell_f(x(t_f))$$

$$V : \; T \times X \to \mathbb{R} \qquad V(t,x) = \inf_{\substack{u(\tau)\in U \\ \tau\in\{t,t+1,\ldots,t_f\}}} J(t,x;u(\cdot))$$

Bellman equation:

$$V(t_f, x) = \ell_f(x) \qquad \text{for all } x \in X$$

$$V(t,x) = \inf_{u\in U}\{\ell(x,u) + V(t+1, \phi(x,u))\}$$

$$\text{for all } x \in X \text{ and all } t \in \{t_0, t_{0+1}, \ldots, t_f - 1\}$$

the cost-to-go

continuous-time systems

(Bellman equation:)

$$V(t,x) = \inf_{\substack{u(\tau)\\ \tau\in[\,t,t+\delta t\,]}} \{\int_t^{t+\delta t} \ell(x(\tau),u(\tau))d\tau + V(t+\delta t, x(t+\delta t))\}$$

the cost-to-go

continuous-time systems

(Bellman equation:)

$$V(t,x) = \inf_{\substack{u(\tau) \\ \tau \in [\,t, t+\delta t\,]}} \{ \int_t^{t+\delta t} \ell(x(\tau), u(\tau)) d\tau + V(t+\delta t, x(t+\delta t)) \}$$

We consider the infinitesimal version ($\delta t \to 0$) of the above relationship

the cost-to-go

continuous-time systems

(Bellman equation:)

$$V(t,x) = \inf_{\substack{u(\tau) \\ \tau \in [\, t, t+\delta t\,]}} \{ \int_t^{t+\delta t} \ell(x(\tau), u(\tau)) d\tau + V(t+\delta t, x(t+\delta t)) \}$$

We consider the infinitesimal version ($\delta t \to 0$) of the above relationship

first-order Taylor expansion:

$$x(t) = x \qquad x(t+\delta t) = x(t) + \delta x$$

$$=$$

$$=$$

$$=$$

the cost-to-go

continuous-time systems

(Bellman equation:)

$$V(t,x) = \inf_{\substack{u(\tau) \\ \tau \in [\,t, t+\delta t\,]}} \{\int_t^{t+\delta t} \ell(x(\tau), u(\tau))d\tau + V(t+\delta t, x(t+\delta t))\}$$

We consider the infinitesimal version ($\delta t \to 0$) of the above relationship

first-order Taylor expansion:

$$x(t) = x \qquad\qquad x(t+\delta t) = x(t) + \delta x$$

$$= x(t) + \frac{dx}{dt}(t)\delta t + o(\delta t)$$

$$=$$

$$=$$

the cost-to-go

continuous-time systems

(Bellman equation:)

$$V(t,x) = \inf_{\substack{u(\tau)\\ \tau\in[\,t,t+\delta t\,]}} \{\int_t^{t+\delta t} \ell(x(\tau),u(\tau))d\tau + V(t+\delta t, x(t+\delta t))\}$$

We consider the infinitesimal version ($\delta t \to 0$) of the above relationship

first-order Taylor expansion:

$$x(t) = x \qquad x(t+\delta t) = x(t) + \delta x$$

$$= x(t) + \frac{dx}{dt}(t)\delta t + o(\delta t)$$

$$= x(t) + f(x(t),u(t))\delta t + o(\delta t)$$

$$=$$

the cost-to-go

continuous-time systems

(Bellman equation:)

$$V(t,x) = \inf_{\substack{u(\tau) \\ \tau\in[\,t,t+\delta t\,]}} \{\int_t^{t+\delta t} \ell(x(\tau),u(\tau))d\tau + V(t+\delta t, x(t+\delta t))\}$$

We consider the infinitesimal version ($\delta t \to 0$) of the above relationship

first-order Taylor expansion:

$$x(t) = x \qquad x(t+\delta t) = x(t) + \delta x$$

$$= x(t) + \frac{dx}{dt}(t)\delta t + o(\delta t)$$

$$= x(t) + f(x(t),u(t))\delta t + o(\delta t)$$

$$= x + \underbrace{f(x,u(t))\delta t + o(\delta t)}_{\delta x}$$

the cost-to-go

continuous-time systems

first-order Taylor expansion:

$$V(t+\delta t, x(t+\delta t)) =$$

$$=$$

the cost-to-go

continuous-time systems

first-order Taylor expansion:

$$\begin{aligned} V(t+\delta t, x(t+\delta t)) = {} & V(t, x(t)) + \frac{\partial V}{\partial t}(t, x(t))\delta t + o(\delta t) \\ & + \left(\frac{\partial V}{\partial x}(t, x(t))\right)^{\mathrm{T}} \delta x + o(\delta x) \\ = {} & \end{aligned}$$

the cost-to-go

continuous-time systems

first-order Taylor expansion:

$$
\begin{aligned}
V(t+\delta t, x(t+\delta t)) &= V(t, x(t)) + \frac{\partial V}{\partial t}(t, x(t))\delta t + o(\delta t) \\
&\qquad + \left(\frac{\partial V}{\partial x}(t, x(t))\right)^{\mathrm{T}} \delta x + o(\delta x) \\
&= V(t, x) + \frac{\partial V}{\partial t}(t, x)\delta t \\
&\qquad + \left(\frac{\partial V}{\partial x}(t, x)\right)^{\mathrm{T}} f(x, u(t))\delta t + o(\delta t)
\end{aligned}
$$

the cost-to-go

continuous-time systems

first-order Taylor expansion:

$$\begin{aligned} V(t+\delta t, x(t+\delta t)) &= V(t, x(t)) + \frac{\partial V}{\partial t}(t, x(t))\delta t + o(\delta t) \\ &\qquad + \left(\frac{\partial V}{\partial x}(t, x(t))\right)^{\mathrm{T}} \delta x + o(\delta x) \\ &= V(t, x) + \frac{\partial V}{\partial t}(t, x)\delta t \\ &\qquad + \left(\frac{\partial V}{\partial x}(t, x)\right)^{\mathrm{T}} f(x, u(t))\delta t + o(\delta t) \end{aligned}$$

$$\begin{aligned} \int_t^{t+\delta t} \ell(x(\tau), u(\tau))d\tau &= \int_0^{\delta t} \ell(x(t+\tau), u(t+\tau))d\tau \\ &= \ell(x(t), u(t))\delta t + o(\delta t) \\ &= \ell(x, u(t))\delta t + o(\delta t) \end{aligned}$$

the cost-to-go

continuous-time systems

$$V(t,x) = \inf_{\substack{u(\tau) \\ \tau \in [\, t, t+\delta t\,]}} \{ \int_t^{t+\delta t} \ell(x(\tau), u(\tau))\, d\tau + V(t+\delta t, x(t+\delta t)) \}$$

the cost-to-go

continuous-time systems

$$V(t,x) = \inf_{\substack{u(\tau) \\ \tau\in[\,t,t+\delta t\,]}} \{\int_t^{t+\delta t} \ell(x(\tau),u(\tau))d\tau + V(t+\delta t, x(t+\delta t))\}$$

plugin all:

the cost-to-go

continuous-time systems

$$V(t,x) = \inf_{\substack{u(\tau)\\ \tau\in[\,t,t+\delta t\,]}} \{ \int_t^{t+\delta t} \ell(x(\tau),u(\tau))d\tau + V(t+\delta t, x(t+\delta t)) \}$$

plugin all:

$$V(t,x) = \inf_{\substack{u(\tau)\\ \tau\in[\,t,t+\delta t\,]}} \{ \ell(x,u(t))\delta t + V(t,x) + \frac{\partial V}{\partial t}(t,x)\delta t + \left(\frac{\partial V}{\partial x}(t,x)\right)^{\mathrm{T}}$$

$$0 = \frac{\partial V}{\partial t}(t,x)\delta t + \inf_{\substack{u(\tau)\\ \tau\in[\,t,t+\delta t\,]}} \{ \ell(x,u(t))\delta t + \left(\frac{\partial V}{\partial x}(t,x)\right)^{\mathrm{T}} f(x,u(t))\delta$$

$$= \frac{\partial V}{\partial t}(t,x)\delta t + \inf_{\substack{u(\tau)\\ \tau\in[\,t,t+\delta t\,]}} \{ \ell(x,u(t))\delta t + \left(\frac{\partial V}{\partial x}(t,x)\right)^{\mathrm{T}} f(x,u(t))\delta$$

the cost-to-go

continuous-time systems

dividing by δt:

$$0 = \frac{\partial V}{\partial t}(t,x) + \inf_{\substack{u(\tau) \\ \tau \in [\,t, t+\delta t\,]}} \{\ell(x,u(t)) + \left(\frac{\partial V}{\partial x}(t,x)\right)^{\mathrm{T}} f(x,u(t)) + \frac{o(\delta t)}{\delta t}\}$$

the cost-to-go

continuous-time systems

dividing by δt:

$$0 = \frac{\partial V}{\partial t}(t, x) + \inf_{\substack{u(\tau) \\ \tau \in [\, t, t+\delta t\,]}} \{\ell(x, u(t)) + \left(\frac{\partial V}{\partial x}(t, x)\right)^{\mathrm{T}} f(x, u(t)) + \frac{o(\delta t)}{\delta t}\}$$

Let us take δt to be small as $\delta t \to 0$.

the cost-to-go

continuous-time systems

dividing by δt:

$$0 = \frac{\partial V}{\partial t}(t,x) + \inf_{\substack{u(\tau) \\ \tau \in [\, t, t+\delta t\,]}} \{\ell(x,u(t)) + \left(\frac{\partial V}{\partial x}(t,x)\right)^{\mathrm{T}} f(x,u(t)) + \frac{o(\delta t)}{\delta t}\}$$

Let us take δt to be small as $\delta t \to 0$. The higher order term $o(\delta t)/\delta t$ disappears, and the infimum is taken over the instantaneous value of u at time t.

the cost-to-go

continuous-time systems

dividing by δt:

$$0 = \frac{\partial V}{\partial t}(t,x) + \inf_{\substack{u(\tau) \\ \tau \in [\, t, t+\delta t \,]}} \{\ell(x,u(t)) + \left(\frac{\partial V}{\partial x}(t,x)\right)^{\mathrm{T}} f(x,u(t)) + \frac{o(\delta t)}{\delta t}\}$$

Let us take δt to be small as $\delta t \to 0$. The higher order term $o(\delta t)/\delta t$ disappears, and the infimum is taken over the instantaneous value of u at time t.

$$0 = \frac{\partial V}{\partial t}(t,x) + \inf_{u \in \mathbb{R}^m} \{\ell(x,u) + \left(\frac{\partial V}{\partial x}(t,x)\right)^{\mathrm{T}} f(x,u)\}$$

Hamilton-Jacobi-Bellman equation

Continuous-time systems

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_\mathrm{f}} \ell(x(\tau), u(\tau))\, d\tau + \ell_\mathrm{f}(x_\mathrm{f}(t_\mathrm{f}))$$

$$V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \qquad\qquad V(t, x) = \inf_{\substack{u(\tau) \\ \tau \in [\, t, t_\mathrm{f}\,]}} J(t, x; u(\cdot))$$

if $t = t_\mathrm{f}$:

$$\begin{aligned} V(t_\mathrm{f}, x) &= \inf_{u(t_\mathrm{f})} J(t_\mathrm{f}, x; u(t_\mathrm{f})) \\ &= \inf_{u(t_\mathrm{f})} \underbrace{\ell_\mathrm{f}(x)}_{\text{independent of } u(t_\mathrm{f})} = \ell_\mathrm{f}(x) \end{aligned}$$

Hamilton-Jacobi-Bellman equation

Continuous-time systems

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_\mathrm{f}} \ell(x(\tau), u(\tau))\, d\tau + \ell_\mathrm{f}(x_\mathrm{f}(t_\mathrm{f}))$$

$$V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \qquad V(t, x) = \inf_{\substack{u(\tau) \\ \tau \in [\, t, t_\mathrm{f}\,]}} J(t, x; u(\cdot))$$

For $t < t_\mathrm{f}$:

$$\begin{aligned} V(t, x) &= \inf_{\substack{u(\tau) \\ \tau \in [\, t, t_\mathrm{f}\,]}} J(t, x; u(\cdot)) \\ &= \inf_{\substack{u(\tau) \\ \tau \in [\, t, t+\delta t\,]}} \left\{ \int_t^{t+\delta t} \ell(x(\tau), u(\tau))\, d\tau + V(t + \delta t, x(t + \delta t)) \right\} \end{aligned}$$

This represents the principle of optimality

Hamilton-Jacobi-Bellman equation

Continuous-time systems

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_\mathrm{f}} \ell(x(\tau), u(\tau))\, d\tau + \ell_\mathrm{f}(x_\mathrm{f}(t_\mathrm{f}))$$

$$V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \qquad\qquad V(t, x) = \inf_{\substack{u(\tau) \\ \tau \in [\, t, t_\mathrm{f}\,]}} J(t, x; u(\cdot))$$

For $t < t_\mathrm{f}$:

$$0 = \frac{\partial V}{\partial t}(t, x) + \inf_{u \in \mathbb{R}^m} \left\{ \ell(x, u) + \left(\frac{\partial V}{\partial x}(t, x) \right)^\mathrm{T} f(x, u) \right\}$$

Hamilton-Jacobi-Bellman equation

Continuous-time systems

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_\mathrm{f}} \ell(x(\tau), u(\tau))\, d\tau + \ell_\mathrm{f}(x_\mathrm{f}(t_\mathrm{f}))$$

$$V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \qquad V(t,x) = \inf_{\substack{u(\tau) \\ \tau \in [\,t, t_\mathrm{f}\,]}} J(t, x; u(\cdot))$$

Hamilton-Jacobi-Bellman equation:

$$V(t_\mathrm{f}, x) = \ell_\mathrm{f}(x) \qquad \text{for all } x \in \mathbb{R}^n$$

$$0 = \frac{\partial V}{\partial t}(t,x) + \inf_{u \in \mathbb{R}^m} \left\{ \ell(x,u) + \left(\frac{\partial V}{\partial x}(t,x) \right)^{\mathrm{T}} f(x,u) \right\}$$

for all $x \in \mathbb{R}^n$ and all $t \in [\, t_0, t_\mathrm{f}\,)$

state feedback implementation

Continuous-time systems

Let $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a solution to

$$V(t_{\mathrm{f}}, x) = \ell_{\mathrm{f}}(x) \qquad \text{for all } x \in \mathbb{R}^n$$

$$0 = \frac{\partial V}{\partial t}(t, x) + \inf_{u \in \mathbb{R}^m} \{\ell(x, u) + \left(\frac{\partial V}{\partial x}(t, x)\right)^{\mathrm{T}} f(x, u)\}$$

$$\text{for all } x \in \mathbb{R}^n \text{ and all } t \in [\, t_0, t_{\mathrm{f}} \,)$$

state feedback implementation

Continuous-time systems

Let $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a solution to

$$V(t_\mathrm{f}, x) = \ell_\mathrm{f}(x) \qquad \text{for all } x \in \mathbb{R}^n$$

$$0 = \frac{\partial V}{\partial t}(t, x) + \inf_{u \in \mathbb{R}^m} \left\{ \ell(x, u) + \left(\frac{\partial V}{\partial x}(t, x) \right)^\mathrm{T} f(x, u) \right\}$$

for all $x \in \mathbb{R}^n$ and all $t \in [\, t_0, t_\mathrm{f} \,)$

State feedback control:

$$u(t) = u(x(t)) = \arg\min_{u \in \mathbb{R}^m} \underbrace{\left\{ \ell(x(t), u) + \left(\frac{\partial V}{\partial x}(t, x(t)) \right)^\mathrm{T} f(x(t), u) \right\}}_{\text{computed using the measured state } x(t)}$$

$$\dot{x}(t) = f(x(t), u(t)) \qquad x(t_0) = x_0$$

continuous-time systems

optimal control problem

$$\dot{x}(t) = f(x(t), u(t)) \qquad x(t_0) = x_0 \qquad t \in [\, t_0, t_\mathrm{f} \,]$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_\mathrm{f}} \ell(x(\tau), u(\tau))\, d\tau + \ell_\mathrm{f}(x_\mathrm{f}(t_\mathrm{f}))$$

$$\inf_{u(\cdot)} J(t_0, x_0; u(\cdot))$$

a solution:

$$0 = \frac{\partial V}{\partial t}(t, x) + \inf_{u \in \mathbb{R}^m} \{\ell(x, u) + \left(\frac{\partial V}{\partial x}(t, x)\right)^\mathrm{T} f(x, u)\}$$

$$V(t_\mathrm{f}, x) = \ell_\mathrm{f}(x) \qquad \text{for all } x \in \mathbb{R}^n$$

$$u(t) = u(x(t)) = \arg\min_{u \in \mathbb{R}^m} \{\ell(x(t), u) + \left(\frac{\partial V}{\partial x}(t, x(t))\right)^\mathrm{T} f(x(t), u)\}$$

contents

optimal control systems

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