Advanced Control Systems Engineering I:

Optimal Control

contents

optimal control systems

- nonlinear dynamical systems and linear approximations
- dynamic programming
- the principle of optimality
- optimal control of finite state systems
- optimal control of discrete-time systems
- optimal control of continuous-time systems
- optimal control of linear systems
- decentralized optimal control
  - decentralization and integration via mechanism design

continuous-time systems

optimal control problem

$$\dot{x}(t) = f(x(t), u(t)) \qquad x(t_0) = x_0 \qquad t \in [\, t_0, t_\mathrm{f} \,]$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

for a given $x(t_0) = x_0 \in \mathbb{R}^n$

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_\mathrm{f}} \ell(x(\tau), u(\tau))\, d\tau + \ell_\mathrm{f}(x_\mathrm{f}(t_\mathrm{f}))$$

optimal control problem

$$\inf_{\substack{u(\tau) \\ \tau \in [\, t_0, t_\mathrm{f} \,]}} J(t_0, x_0; u(\cdot))$$

the cost-to-go

continuous-time systems

for a given $x(t_0) = x_0 \in \mathbb{R}^n$

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_\mathrm{f}} \ell(x(\tau), u(\tau))\, d\tau + \ell_\mathrm{f}(x_\mathrm{f}(t_\mathrm{f}))$$

optimal control problem

$$\inf_{\substack{u(\tau) \\ \tau \in [\, t_0, t_\mathrm{f}\,]}} \quad J(t_0, x_0; u(\cdot))$$

define the cost-to-go

$$V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \qquad\qquad V(t, x) = \inf_{\substack{u(\tau) \\ \tau \in [\, t, t_\mathrm{f}\,]}} \quad J(t, x; u(\cdot))$$

computing the cost-to-go $V(t_0, x_0)$ from the initial state x_0 at the initial time t_0 essentially amounts to minimize the cost $J(t_0, x_0; u(\cdot))$.

the cost-to-go

continuous-time systems

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_f} \ell(x(\tau), u(\tau))\, d\tau + \ell_f(x_f(t_f))$$

$$V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \qquad V(t,x) = \inf_{\substack{u(\tau) \\ \tau \in [\, t, t_f \,]}} J(t, x; u(\cdot))$$

For $t < t_f$, let us pick some small positive constant δt so that $t + \delta t$ is still smaller than t_f:

$$\begin{aligned} V(t,x) &= \inf_{\substack{u(\tau) \\ \tau \in [\, t, t_f \,]}} J(t, x; u(\cdot)) \\ &= \inf_{\substack{u(\tau) \\ \tau \in [\, t, t+\delta t \,]}} \left\{ \int_t^{t+\delta t} \ell(x(\tau), u(\tau))\, d\tau + V(t+\delta t, x(t+\delta t)) \right\} \end{aligned}$$

This represents the principle of optimality

Hamilton-Jacobi-Bellman equation

continuous-time systems

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_f} \ell(x(\tau), u(\tau))\, d\tau + \ell_f(x_f(t_f))$$

$$V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \qquad\qquad V(t, x) = \inf_{\substack{u(\tau) \\ \tau \in [\, t, t_f \,]}} J(t, x; u(\cdot))$$

Hamilton-Jacobi-Bellman equation:

$V(t_f, x) = \ell_f(x)$ for all $x \in \mathbb{R}^n$

$$0 = \frac{\partial V}{\partial t}(t, x) + \inf_{u \in \mathbb{R}^m} \{\ell(x, u) + \left(\frac{\partial V}{\partial x}(t, x)\right)^{\mathrm{T}} f(x, u)\}$$

for all $x \in \mathbb{R}^n$ and all $t \in [\, t_0, t_f \,)$

state feedback implementation

continuous-time systems

Let $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a solution to

$$V(t_\mathrm{f}, x) = \ell_\mathrm{f}(x) \qquad \text{for all } x \in \mathbb{R}^n$$

$$0 = \frac{\partial V}{\partial t}(t, x) + \inf_{u \in \mathbb{R}^m} \{\ell(x, u) + \left(\frac{\partial V}{\partial x}(t, x)\right)^\mathrm{T} f(x, u)\}$$

for all $x \in \mathbb{R}^n$ and all $t \in [\, t_0, t_\mathrm{f}\,)$

state feedback control:

$$u(t) = u(x(t))$$

$$= \arg\min_{u \in \mathbb{R}^m} \underbrace{\{\ell(x(t), u) + \left(\frac{\partial V}{\partial x}(t, x(t))\right)^\mathrm{T} f(x(t), u)\}}_{\text{computed using the measured state } x(t)}$$

$$\dot{x}(t) = f(x(t), u(t)) \qquad x(t_0) = x_0$$

example

continuous-time systems

$$\dot{x}(t) = f(x(t), u(t)) \qquad x(t_0) = x_0 \qquad t \in [\, t_0, t_\mathrm{f} \,]$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_\mathrm{f}} \ell(x(\tau), u(\tau))\, d\tau + \ell_\mathrm{f}(x_\mathrm{f}(t_\mathrm{f}))$$

example

continuous-time systems

$$\dot{x}(t) = f(x(t), u(t)) \qquad x(t_0) = x_0 \qquad t \in [\, t_0, t_\mathrm{f} \,]$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

$$\dot{x}(t) = u(t) \qquad x(t_0) = x_0 \qquad t \in [\, t_0, t_\mathrm{f} \,]$$

$$x(t) \in \mathbb{R} \qquad u(t) \in \mathbb{R}$$

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_\mathrm{f}} \ell(x(\tau), u(\tau))\, d\tau + \ell_\mathrm{f}(x_\mathrm{f}(t_\mathrm{f}))$$

example

continuous-time systems

$$\dot{x}(t) = f(x(t), u(t)) \qquad x(t_0) = x_0 \qquad t \in [\, t_0, t_\mathrm{f} \,]$$
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$$x(t) \in \mathbb{R} \qquad u(t) \in \mathbb{R}$$

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_\mathrm{f}} \ell(x(\tau), u(\tau))\, d\tau + \ell_\mathrm{f}(x_\mathrm{f}(t_\mathrm{f}))$$

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_\mathrm{f}} x^4(\tau) + u^4(\tau)\, d\tau + \ell_\mathrm{f}(x_\mathrm{f}(t_\mathrm{f}))$$

example

continuous-time systems

$$V(t_{\mathrm{f}}, x) = \ell_{\mathrm{f}}(x) \qquad \text{for all } x \in \mathbb{R}^n$$

$$0 = \frac{\partial V}{\partial t}(t, x) + \inf_{u \in \mathbb{R}^m} \left\{ \ell(x, u) + \left(\frac{\partial V}{\partial x}(t, x) \right)^{\mathrm{T}} f(x, u) \right\}$$

$$\text{for all } x \in \mathbb{R}^n \text{ and all } t \in [\, t_0, t_{\mathrm{f}} \,)$$

example

continuous-time systems

$$V(t_\mathrm{f}, x) = \ell_\mathrm{f}(x) \qquad \text{for all } x \in \mathbb{R}^n$$

$$0 = \frac{\partial V}{\partial t}(t,x) + \inf_{u \in \mathbb{R}^m} \left\{ \ell(x,u) + \left(\frac{\partial V}{\partial x}(t,x) \right)^\mathrm{T} f(x,u) \right\}$$

$$\text{for all } x \in \mathbb{R}^n \text{ and all } t \in [\, t_0, t_\mathrm{f} \,)$$

$$0 = \frac{\partial V}{\partial t}(t,x) + \inf_{u \in \mathbb{R}} \left\{ x^4 + u^4 + \frac{\partial V}{\partial x}(t,x) u \right\}$$

example

continuous-time systems

$$V(t_{\mathrm{f}}, x) = \ell_{\mathrm{f}}(x) \qquad \text{for all } x \in \mathbb{R}^n$$

$$0 = \frac{\partial V}{\partial t}(t, x) + \inf_{u \in \mathbb{R}^m} \{\ell(x, u) + \left(\frac{\partial V}{\partial x}(t, x)\right)^{\mathrm{T}} f(x, u)\}$$

for all $x \in \mathbb{R}^n$ and all $t \in [\, t_0, t_{\mathrm{f}} \,)$

$$0 = \frac{\partial V}{\partial t}(t, x) + x^4 + \inf_{u \in \mathbb{R}} \{u^4 + \frac{\partial V}{\partial x}(t, x) u\}$$

example

continuous-time systems

$$V(t_{\mathrm{f}}, x) = \ell_{\mathrm{f}}(x) \qquad \text{for all } x \in \mathbb{R}^n$$

$$0 = \frac{\partial V}{\partial t}(t, x) + \inf_{u \in \mathbb{R}^m} \{\ell(x, u) + \left(\frac{\partial V}{\partial x}(t, x)\right)^{\mathrm{T}} f(x, u)\}$$

for all $x \in \mathbb{R}^n$ and all $t \in [\, t_0, t_{\mathrm{f}} \,)$

$$0 = \frac{\partial V}{\partial t}(t, x) + x^4 + \inf_{u \in \mathbb{R}} \{u^4 + \frac{\partial V}{\partial x}(t, x) u\}$$

$$u^* = \left(-\frac{1}{4}\frac{\partial V}{\partial x}(t, x)\right)^{1/3} = -\left(\frac{1}{4}\frac{\partial V}{\partial x}(t, x)\right)^{1/3}$$

example

continuous-time systems

$$V(t_\mathrm{f}, x) = \ell_\mathrm{f}(x) \qquad \text{for all } x \in \mathbb{R}^n$$

$$0 = \frac{\partial V}{\partial t}(t,x) + \inf_{u \in \mathbb{R}^m} \{\ell(x,u) + \left(\frac{\partial V}{\partial x}(t,x)\right)^\mathrm{T} f(x,u)\}$$

$$\text{for all } x \in \mathbb{R}^n \text{ and all } t \in [\, t_0, t_\mathrm{f} \,)$$

$$0 = \frac{\partial V}{\partial t}(t,x) + x^4 + \inf_{u \in \mathbb{R}} \{u^4 + \frac{\partial V}{\partial x}(t,x) u\}$$

$$u^* = \left(-\frac{1}{4}\frac{\partial V}{\partial x}(t,x)\right)^{1/3} = -\left(\frac{1}{4}\frac{\partial V}{\partial x}(t,x)\right)^{1/3}$$

$$0 = \frac{\partial V}{\partial t}(t,x) + x^4 + \left(\frac{1}{4}\frac{\partial V}{\partial x}(t,x)\right)^{4/3} - \frac{\partial V}{\partial x}(t,x)\left(\frac{1}{4}\frac{\partial V}{\partial x}(t,x)\right)^{1/3}$$

example

continuous-time systems

$$
\begin{aligned}
V(t_{\mathrm{f}}, x) &= \ell_{\mathrm{f}}(x) \\
0 &= \frac{\partial V}{\partial t}(t, x) + x^4 - 3\left(\frac{1}{4}\frac{\partial V}{\partial x}(t, x)\right)^{4/3}
\end{aligned}
$$

infinite horizon problem

continuous-time systems

$$\dot{x}(t) = f(x(t), u(t)) \qquad x(t_0) = x_0 \qquad t \in [\, t_0, t_\mathrm{f} \,]$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

for a given $x(t_0) = x_0 \in \mathbb{R}^n$

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_\mathrm{f}} \ell(x(\tau), u(\tau))\, d\tau + \ell_\mathrm{f}(x_\mathrm{f}(t_\mathrm{f}))$$

optimal control problem

$$\inf_{\substack{u(\tau) \\ \tau \in [\, t_0, t_\mathrm{f} \,]}} J(t_0, x_0; u(\cdot))$$

infinite horizon problem

continuous-time systems

$$\dot{x}(t) = f(x(t), u(t)) \qquad x(0) = x_0 \qquad t \in [\,0, \infty\,)$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

for a given $x(t_0) = x_0 \in \mathbb{R}^n$

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_\mathrm{f}} \ell(x(\tau), u(\tau))\, d\tau + \ell_\mathrm{f}(x_\mathrm{f}(t_\mathrm{f}))$$

optimal control problem

$$\inf_{\substack{u(\tau) \\ \tau \in [\,t_0, t_\mathrm{f}\,]}} \quad J(t_0, x_0; u(\cdot))$$

infinite horizon problem

continuous-time systems

$$\dot{x}(t) = f(x(t), u(t)) \qquad x(0) = x_0 \qquad t \in [\,0, \infty\,)$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

for a given $x(0) = x_0 \in \mathbb{R}^n$

$$J(x_0; u(\cdot)) = \int_0^\infty \ell(x(\tau), u(\tau))\, d\tau = \lim_{t \to \infty} \int_0^t \ell(x(\tau), u(\tau))\, d\tau$$

optimal control problem

$$\inf_{\substack{u(\tau) \\ \tau \in [\,t_0, t_f\,]}} \quad J(t_0, x_0; u(\cdot))$$

infinite horizon problem

continuous-time systems

$$\dot{x}(t) = f(x(t), u(t)) \qquad x(0) = x_0 \qquad t \in [\,0, \infty\,)$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

for a given $x(0) = x_0 \in \mathbb{R}^n$

$$J(x_0; u(\cdot)) = \int_0^\infty \ell(x(\tau), u(\tau))\, d\tau = \lim_{t \to \infty} \int_0^t \ell(x(\tau), u(\tau))\, d\tau$$

optimal control problem

$$\inf_{\substack{u(\tau) \\ \tau \in [\,0, \infty\,)}} \quad J(x_0; u(\cdot))$$

Hamilton-Jacobi-Bellman equation

infinite horizon problem

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_\mathrm{f}} \ell(x(\tau), u(\tau))\, d\tau + \ell_\mathrm{f}(x_\mathrm{f}(t_\mathrm{f}))$$

$$V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \qquad\qquad V(t, x) = \inf_{\substack{u(\tau) \\ \tau \in [\, t, t_\mathrm{f}\,]}} J(t, x; u(\cdot))$$

Hamilton-Jacobi-Bellman equation:

$$V(t_\mathrm{f}, x) = \ell_\mathrm{f}(x) \qquad \text{for all } x \in \mathbb{R}^n$$

$$0 = \frac{\partial V}{\partial t}(t, x) + \inf_{u \in \mathbb{R}^m} \left\{ \ell(x, u) + \left(\frac{\partial V}{\partial x}(t, x) \right)^\mathrm{T} f(x, u) \right\}$$

for all $x \in \mathbb{R}^n$ and all $t \in [\, t_0, t_\mathrm{f}\,)$

Hamilton-Jacobi-Bellman equation

infinite horizon problem

$$J(x_0; u(\cdot)) = \int_0^{\infty} \ell(x(\tau), u(\tau))\, d\tau$$

$$V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \qquad\qquad V(t,x) = \inf_{\substack{u(\tau) \\ \tau \in [\, t, t_\mathrm{f}\,]}} J(t, x; u(\cdot))$$

Hamilton-Jacobi-Bellman equation:

$$V(t_\mathrm{f}, x) = \ell_\mathrm{f}(x) \qquad \text{for all } x \in \mathbb{R}^n$$

$$0 = \frac{\partial V}{\partial t}(t,x) + \inf_{u \in \mathbb{R}^m} \left\{ \ell(x,u) + \left(\frac{\partial V}{\partial x}(t,x) \right)^{\mathrm{T}} f(x,u) \right\}$$

for all $x \in \mathbb{R}^n$ and all $t \in [\, t_0, t_\mathrm{f}\,)$

Hamilton-Jacobi-Bellman equation

infinite horizon problem

$$J(x_0; u(\cdot)) = \int_0^{\infty} \ell(x(\tau), u(\tau))\, d\tau$$

$$V : \mathbb{R}^n \to \mathbb{R} \qquad\qquad V(x) = \inf_{\substack{u(\tau) \\ \tau \in [\,0, \infty\,)}} J(x; u(\cdot))$$

Hamilton-Jacobi-Bellman equation:

$$V(t_\mathrm{f}, x) = \ell_\mathrm{f}(x) \qquad \text{for all } x \in \mathbb{R}^n$$

$$0 = \frac{\partial V}{\partial t}(t, x) + \inf_{u \in \mathbb{R}^m} \{\ell(x, u) + \left(\frac{\partial V}{\partial x}(t, x)\right)^\mathrm{T} f(x, u)\}$$

for all $x \in \mathbb{R}^n$ and all $t \in [\, t_0, t_\mathrm{f}\,)$

Hamilton-Jacobi-Bellman equation

infinite horizon problem

$$J(x_0; u(\cdot)) = \int_0^\infty \ell(x(\tau), u(\tau))\, d\tau$$

$$V : \mathbb{R}^n \to \mathbb{R} \qquad\qquad V(x) = \inf_{\substack{u(\tau) \\ \tau \in [\,0, \infty\,)}} J(x; u(\cdot))$$

Hamilton-Jacobi-Bellman equation:

$$0 = \inf_{u \in \mathbb{R}^m} \left\{ \ell(x, u) + \left(\frac{\partial V}{\partial x}(x) \right)^{\mathrm{T}} f(x, u) \right\} \qquad \text{for all } x \in \mathbb{R}^n$$

state feedback implementation

Let $V : \mathbb{R}^n \to \mathbb{R}$ be a solution to

$$0 = \inf_{u \in \mathbb{R}^m} \{\ell(x, u) + \left(\frac{\partial V}{\partial x}(x)\right)^{\mathrm{T}} f(x, u)\} \qquad \text{for all } x \in \mathbb{R}^n$$

state feedback control:

$$\begin{aligned} u(t) &= u(x(t)) \\ &= \arg\min_{u \in \mathbb{R}^m} \underbrace{\{\ell(x(t), u) + \left(\frac{\partial V}{\partial x}(x(t))\right)^{\mathrm{T}} f(x(t), u)\}}_{\text{computed using the measured state } x(t)} \end{aligned}$$

$$\dot{x}(t) = f(x(t), u(t)) \qquad x(t_0) = x_0$$

example

infinite horizon problem

$$\dot{x}(t) = u(t) \qquad x(t_0) = x_0 \qquad t \in [\, t_0, t_\mathrm{f} \,]$$
$$x(t) \in \mathbb{R} \qquad u(t) \in \mathbb{R}$$

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{t_\mathrm{f}} x^4(\tau) + u^4(\tau) d\tau + \ell_\mathrm{f}(x_\mathrm{f}(t_\mathrm{f}))$$

$$u^* = -\left(\frac{1}{4}\frac{\partial V}{\partial x}(t, x)\right)^{1/3}$$
$$0 = \frac{\partial V}{\partial t}(t, x) + x^4 - 3\left(\frac{1}{4}\frac{\partial V}{\partial x}(t, x)\right)^{4/3}$$

example

infinite horizon problem

$$\dot{x}(t) = u(t) \qquad x(0) = x_0 \qquad t \in [\,0, \infty\,)$$
$$x(t) \in \mathbb{R} \qquad u(t) \in \mathbb{R}$$

$$J(x_0; u(\cdot)) = \int_0^\infty x^4(\tau) + u^4(\tau)\, d\tau$$

$$u^* = -\left(\frac{1}{4}\frac{\partial V}{\partial x}(x)\right)^{1/3}$$
$$0 = x^4 - 3\left(\frac{1}{4}\frac{\partial V}{\partial x}(x)\right)^{4/3}$$

example

infinite horizon problem

$$\dot{x}(t) = u(t) \qquad x(0) = x_0 \qquad t \in [\,0, \infty\,)$$

$$x(t) \in \mathbb{R} \qquad u(t) \in \mathbb{R}$$

$$J(x_0; u(\cdot)) = \int_0^\infty x^4(\tau) + u^4(\tau)\, d\tau$$

$$u^* = -\left(\frac{1}{4}\frac{\partial V}{\partial x}(x)\right)^{1/3} \qquad u^* = -\left(\frac{1}{3}\right)^{1/4} x$$

$$0 = x^4 - 3\left(\frac{1}{4}\frac{\partial V}{\partial x}(x)\right)^{4/3} \qquad \frac{\partial V}{\partial x}(x) = 4\left(\frac{1}{3}\right)^{3/4} x^3$$

example

infinite horizon problem

$$\dot{x}(t) = u(t) \qquad x(0) = x_0 \qquad t \in [\,0, \infty\,)$$

$$x(t) \in \mathbb{R} \qquad u(t) \in \mathbb{R}$$

$$J(x_0; u(\cdot)) = \int_0^\infty x^4(\tau) + u^4(\tau)\, d\tau \qquad x_0 = 2$$

optimal control $\quad u^* = -\left(\frac{1}{3}\right)^{1/4} x$

example

infinite horizon problem

$$\dot{x}(t) = u(t) \qquad x(0) = x_0 \qquad t \in [\,0, \infty\,)$$

$$x(t) \in \mathbb{R} \qquad u(t) \in \mathbb{R}$$

$$J(x_0; u(\cdot)) = \int_0^\infty x^4(\tau) + u^4(\tau) d\tau \qquad x_0 = 2$$

optimal control $u^* = -\left(\frac{1}{3}\right)^{1/4} x$ $\qquad u = -x^2$

example

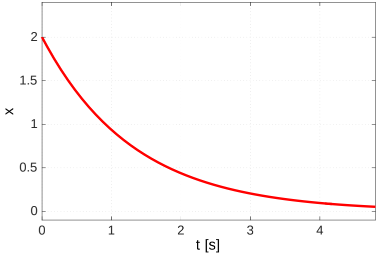
infinite horizon problem

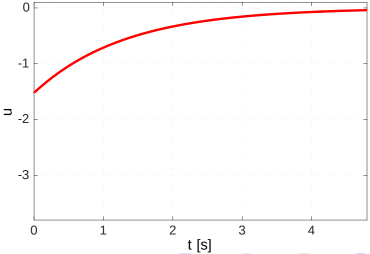
$$\dot{x}(t) = u(t) \qquad x(0) = x_0 \qquad t \in [\,0, \infty\,)$$

$$x(t) \in \mathbb{R} \qquad u(t) \in \mathbb{R}$$

$$J(x_0; u(\cdot)) = \int_0^\infty x^4(\tau) + u^4(\tau) d\tau \qquad x_0 = 2$$

optimal control $\quad u^* = -\left(\frac{1}{3}\right)^{1/4} x \qquad\qquad u = -x^2$





example

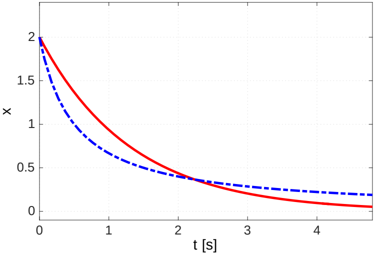
infinite horizon problem

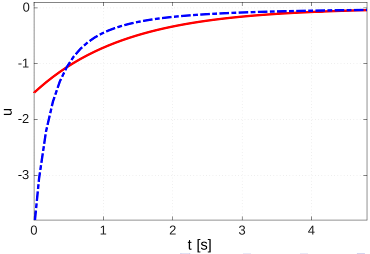
$$\dot{x}(t) = u(t) \qquad x(0) = x_0 \qquad t \in [\,0, \infty\,)$$

$$x(t) \in \mathbb{R} \qquad u(t) \in \mathbb{R}$$

$$J(x_0; u(\cdot)) = \int_0^\infty x^4(\tau) + u^4(\tau) d\tau \qquad x_0 = 2$$

optimal control $u^* = -\left(\frac{1}{3}\right)^{1/4} x$ $\qquad u = -x^2$





example

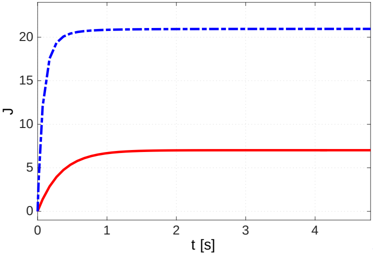
infinite horizon problem

$$\dot{x}(t) = u(t) \qquad x(0) = x_0 \qquad t \in [\,0, \infty\,)$$

$$x(t) \in \mathbb{R} \qquad u(t) \in \mathbb{R}$$

$$J(x_0; u(\cdot)) = \int_0^\infty x^4(\tau) + u^4(\tau)\, d\tau \qquad x_0 = 2$$

optimal control $\quad u^* = -\left(\frac{1}{3}\right)^{1/4} x \qquad\qquad u = -x^2$



example 02

infinite horizon problem

$$\dot{x}(t) = x^2(t) + u(t) \qquad x(0) = x_0 \qquad t \in [\,0, \infty\,)$$

$$x(t) \in \mathbb{R} \qquad u(t) \in \mathbb{R}$$

$$J(x_0; u(\cdot)) = \int_0^\infty x^2(\tau) + u^2(\tau) d\tau$$

$$0 = \inf_{u \in \mathbb{R}} \{x^2 + u^2 + \frac{\partial V}{\partial x}(x)(x^2 + u)\}$$

example 02

infinite horizon problem

$$\dot{x}(t) = x^2(t) + u(t) \qquad x(0) = x_0 \qquad t \in [\,0, \infty\,)$$

$$x(t) \in \mathbb{R} \qquad u(t) \in \mathbb{R}$$

$$J(x_0; u(\cdot)) = \int_0^\infty x^2(\tau) + u^2(\tau)\, d\tau$$

$$0 = x^2 + \frac{\partial V}{\partial x}(x) x^2 + \inf_{u \in \mathbb{R}} \{ u^2 + \frac{\partial V}{\partial x}(x) u \}$$

example 02

infinite horizon problem

$$\dot{x}(t) = x^2(t) + u(t) \qquad x(0) = x_0 \qquad t \in [\,0, \infty\,)$$

$$x(t) \in \mathbb{R} \qquad u(t) \in \mathbb{R}$$

$$J(x_0; u(\cdot)) = \int_0^\infty x^2(\tau) + u^2(\tau)\,d\tau$$

$$0 = x^2 + \frac{\partial V}{\partial x}(x)x^2 + \inf_{u \in \mathbb{R}} \{u^2 + \frac{\partial V}{\partial x}(x)u\}$$

$$u^* = -\frac{1}{2}\frac{\partial V}{\partial x}(x)$$

$$0 = -\frac{1}{4}\left(\frac{\partial V}{\partial x}(x)\right)^2 + x^2\frac{\partial V}{\partial x}(x) + x^2$$

example 02

infinite horizon problem

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closed-loop system

$$\begin{aligned}\dot{x}(t) &= x^2(t) + u(t) \\ &= x(t)\sqrt{x^2(t)+1} \qquad \text{or} \qquad = -x(t)\sqrt{x^2(t)+1}\end{aligned}$$

unstable | stable

example 02

infinite horizon problem

$$\dot{x}(t) = x^2(t) + u(t) \qquad x(0) = x_0 \qquad t \in [\,0, \infty\,)$$

$$x(t) \in \mathbb{R} \qquad u(t) \in \mathbb{R}$$

$$J(x_0; u(\cdot)) = \int_0^\infty x^2(\tau) + u^2(\tau) d\tau \qquad x_0 = 6$$

optimal control $\quad u^* = -x^2 - x\sqrt{x^2 + 1}$

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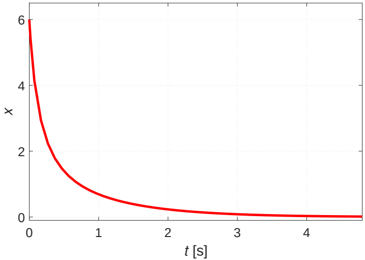

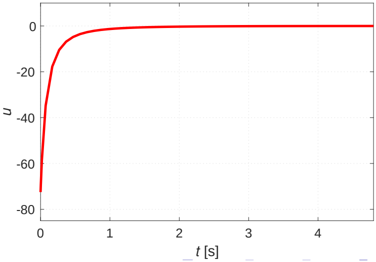

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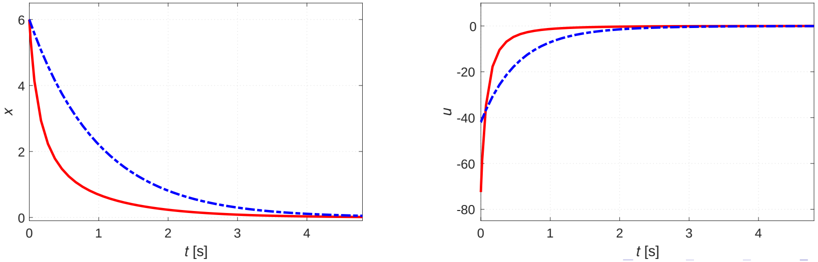

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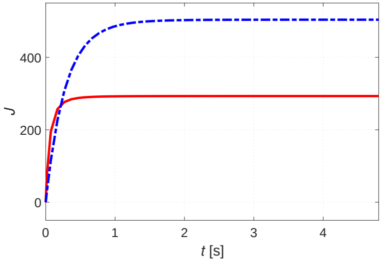

contents

optimal control systems

- nonlinear dynamical systems and linear approximations
- dynamic programming
- the principle of optimality
- optimal control of finite state systems
- optimal control of discrete-time systems
- optimal control of continuous-time systems
- optimal control of linear systems
- decentralized optimal control
  - decentralization and integration via mechanism design