Advanced Control Systems Engineering I:

Optimal Control

contents

optimal control systems

- nonlinear dynamical systems and linear approximations
- dynamic programming
- the principle of optimality
- optimal control of finite state systems
- optimal control of discrete-time systems
- optimal control of continuous-time systems
- optimal control of linear systems
- decentralized optimal control
  - decentralization and integration via mechanism design

infinite horizon problem

continuous-time systems

$$\dot{x}(t) = f(x(t), u(t)) \qquad x(0) = x_0 \qquad t \in [\,0, \infty\,)$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

for a given $x(0) = x_0 \in \mathbb{R}^n$

$$J(x_0; u(\cdot)) = \int_0^\infty \ell(x(\tau), u(\tau))\, d\tau$$

optimal control problem

$$\inf_{\substack{u(\tau) \\ \tau \in [\,0, \infty\,)}} J(x_0; u(\cdot))$$

Hamilton-Jacobi-Bellman equation

infinite horizon problem

Hamilton-Jacobi-Bellman equation:

$$0 = \inf_{u \in \mathbb{R}^m} \left\{ \ell(x,u) + \left(\frac{\partial V}{\partial x}(x) \right)^{\mathrm{T}} f(x,u) \right\} \qquad \text{for all } x \in \mathbb{R}^n$$

Let $V : \mathbb{R}^n \to \mathbb{R}$ be a solution to HJB equation.

state feedback control:

$$\begin{aligned} u(t) &= u(x(t)) \\ &= \arg\min_{u \in \mathbb{R}^m} \underbrace{\left\{ \ell(x(t),u) + \left(\frac{\partial V}{\partial x}(x(t)) \right)^{\mathrm{T}} f(x(t),u) \right\}}_{\text{computed using the measured state } x(t)} \end{aligned}$$

$$\dot{x}(t) = f(x(t), u(t)) \qquad x(t_0) = x_0$$

linear quadratic regulator problem

$$\dot{x}(t) = f(x(t), u(t)) \qquad x(0) = x_0 \qquad t \in [\,0, \infty\,)$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

for a given $x(0) = x_0 \in \mathbb{R}^n$

$$J(x_0; u(\cdot)) = \int_0^\infty \ell(x(\tau), u(\tau))\, d\tau$$

optimal control problem

$$\inf_{\substack{u(\tau) \\ \tau \in [\,0, \infty\,)}} J(x_0; u(\cdot))$$

linear quadratic regulator problem

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x(0) = x_0 \qquad t \in [\,0, \infty\,)$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

$$A \in \mathbb{R}^{n\times n} \qquad B \in \mathbb{R}^{n\times m}$$

for a given $x(0) = x_0 \in \mathbb{R}^n$

$$J(x_0; u(\cdot)) = \int_0^\infty \ell(x(\tau), u(\tau))\, d\tau$$

optimal control problem

$$\inf_{\substack{u(\tau) \\ \tau\in[\,0,\infty\,)}} J(x_0; u(\cdot))$$

linear quadratic regulator problem

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x(0) = x_0 \qquad t \in [\,0, \infty\,)$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

$$A \in \mathbb{R}^{n \times n} \qquad B \in \mathbb{R}^{n \times m}$$

for a given $x(0) = x_0 \in \mathbb{R}^n$

$$J(x_0; u(\cdot)) = \int_0^\infty x^{\mathrm{T}}(\tau)\,Qx(\tau) + u^{\mathrm{T}}(\tau)\,Ru(\tau)\,d\tau$$

$$Q \in \mathbb{R}^{n \times n}, Q = Q^{\mathrm{T}} \geq 0 \qquad R \in \mathbb{R}^{m \times m}, R = R^{\mathrm{T}} > 0$$

optimal control problem

$$\inf_{\substack{u(\tau) \\ \tau \in [\,0, \infty\,)}} \quad J(x_0; u(\cdot))$$

Hamilton-Jacobi-Bellman equation

linear quadratic regulator problem

Hamilton-Jacobi-Bellman equation:

$$0 = \inf_{u \in \mathbb{R}^m} \{\ell(x, u) + \left(\frac{\partial V}{\partial x}(x)\right)^{\mathrm{T}} f(x, u)\} \qquad \text{for all } x \in \mathbb{R}^n$$

Hamilton-Jacobi-Bellman equation

linear quadratic regulator problem

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$$0 = \inf_{u \in \mathbb{R}^m} \{x^{\mathrm{T}} Q x + u^{\mathrm{T}} R u + \left(\frac{\partial V}{\partial x}(x)\right)^{\mathrm{T}} (Ax + Bu)\}$$

Hamilton-Jacobi-Bellman equation

linear quadratic regulator problem

Hamilton-Jacobi-Bellman equation:

$$0 = \inf_{u \in \mathbb{R}^m} \{\ell(x,u) + \left(\frac{\partial V}{\partial x}(x)\right)^{\mathrm{T}} f(x,u)\} \qquad \text{for all } x \in \mathbb{R}^n$$

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$$0 = x^{\mathrm{T}} Q x + \left(\frac{\partial V}{\partial x}(x)\right)^{\mathrm{T}} Ax + \inf_{u \in \mathbb{R}^m} \{u^{\mathrm{T}} R u + \left(\frac{\partial V}{\partial x}(x)\right)^{\mathrm{T}} Bu\}$$

Hamilton-Jacobi-Bellman equation

linear quadratic regulator problem

Hamilton-Jacobi-Bellman equation:

$$0 = \inf_{u \in \mathbb{R}^m} \{\ell(x,u) + \left(\frac{\partial V}{\partial x}(x)\right)^{\mathrm{T}} f(x,u)\} \qquad \text{for all } x \in \mathbb{R}^n$$

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$$0 = x^{\mathrm{T}} Q x + \left(\frac{\partial V}{\partial x}(x)\right)^{\mathrm{T}} Ax + \inf_{u \in \mathbb{R}^m} \{u^{\mathrm{T}} R u + \left(\frac{\partial V}{\partial x}(x)\right)^{\mathrm{T}} Bu\}$$

$$2Ru + B^{\mathrm{T}} \frac{\partial V}{\partial x}(x) = 0 \quad \Rightarrow \quad u^* = -\frac{1}{2} R^{-1} B^{\mathrm{T}} \frac{\partial V}{\partial x}(x)$$

Riccati equation

linear quadratic regulator problem

$$0 = x^{\mathrm{T}} Q x + \left(\frac{\partial V}{\partial x}(x) \right)^{\mathrm{T}} A x - \left(-\frac{1}{2} R^{-1} B^{\mathrm{T}} \frac{\partial V}{\partial x}(x) \right)^{\mathrm{T}} R \left(-\frac{1}{2} R^{-1} B^{\mathrm{T}} \frac{\partial V}{\partial x}$$

Riccati equation

linear quadratic regulator problem

$$0 = x^{\mathrm{T}} Qx + \left(\frac{\partial V}{\partial x}(x) \right)^{\mathrm{T}} Ax - \left(-\frac{1}{2} R^{-1} B^{\mathrm{T}} \frac{\partial V}{\partial x}(x) \right)^{\mathrm{T}} R \left(-\frac{1}{2} R^{-1} B^{\mathrm{T}} \frac{\partial V}{\partial x}\right.$$

For LQ problem, the cost-to-go is given as quadratic function of x, cf. [problem 2.3-1, Optimal Control, 1990.]

$$V(x) = x^{\mathrm{T}} Px \quad P = P^{\mathrm{T}} > 0 \qquad \frac{\partial V}{\partial x}(x) = 2Px$$

plug-in:

Riccati equation

linear quadratic regulator problem

$$0 = x^{\mathrm{T}} Q x + \left(\frac{\partial V}{\partial x}(x) \right)^{\mathrm{T}} A x - \left(-\frac{1}{2} R^{-1} B^{\mathrm{T}} \frac{\partial V}{\partial x}(x) \right)^{\mathrm{T}} R \left(-\frac{1}{2} R^{-1} B^{\mathrm{T}} \frac{\partial V}{\partial x}\right.$$

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$$V(x) = x^{\mathrm{T}} P x \quad P = P^{\mathrm{T}} > 0 \qquad \frac{\partial V}{\partial x}(x) = 2Px$$

plug-in:

$$0 = x^{\mathrm{T}} Q x + 2x^{\mathrm{T}} P A x - (-R^{-1} B^{\mathrm{T}} P x)^{\mathrm{T}} R (-R^{-1} B^{\mathrm{T}} P x)$$

Riccati equation

linear quadratic regulator problem

$$0 = x^{\mathrm{T}} Q x + \left(\frac{\partial V}{\partial x}(x) \right)^{\mathrm{T}} A x - \left(-\frac{1}{2} R^{-1} B^{\mathrm{T}} \frac{\partial V}{\partial x}(x) \right)^{\mathrm{T}} R \left(-\frac{1}{2} R^{-1} B^{\mathrm{T}} \frac{\partial V}{\partial x}\right.$$

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$$V(x) = x^{\mathrm{T}} P x \quad P = P^{\mathrm{T}} > 0 \qquad\qquad \frac{\partial V}{\partial x}(x) = 2Px$$

plug-in:

$$0 = x^{\mathrm{T}} Q x + 2 x^{\mathrm{T}} P A x - (-R^{-1} B^{\mathrm{T}} P x)^{\mathrm{T}} R (-R^{-1} B^{\mathrm{T}} P x)$$

$$0 = x^{\mathrm{T}} (PA + A^{\mathrm{T}} P - PBR^{-1} B^{\mathrm{T}} P + Q) x \qquad \text{for all } x \in \mathbb{R}^n$$

Riccati equation

linear quadratic regulator problem

$$0 = x^{\mathrm{T}} Q x + \left(\frac{\partial V}{\partial x}(x) \right)^{\mathrm{T}} A x - \left(-\frac{1}{2} R^{-1} B^{\mathrm{T}} \frac{\partial V}{\partial x}(x) \right)^{\mathrm{T}} R \left(-\frac{1}{2} R^{-1} B^{\mathrm{T}} \frac{\partial V}{\partial x}\right.$$

For LQ problem, the cost-to-go is given as quadratic function of x, cf. [problem 2.3-1, Optimal Control, 1990.]

$$V(x) = x^{\mathrm{T}} P x \quad P = P^{\mathrm{T}} > 0 \qquad \frac{\partial V}{\partial x}(x) = 2Px$$

plug-in:

$$0 = x^{\mathrm{T}} Q x + 2 x^{\mathrm{T}} P A x - (-R^{-1} B^{\mathrm{T}} P x)^{\mathrm{T}} R (-R^{-1} B^{\mathrm{T}} P x)$$

$$0 = x^{\mathrm{T}} (PA + A^{\mathrm{T}} P - P B R^{-1} B^{\mathrm{T}} P + Q) x \qquad \text{for all } x \in \mathbb{R}^n$$

Riccati equation:

$$0 = PA + A^{\mathrm{T}} P - P B R^{-1} B^{\mathrm{T}} P + Q$$

linear quadratic regulator problem

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x(0) = x_0 \qquad t \in [\,0, \infty\,)$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

$$A \in \mathbb{R}^{n \times n} \qquad B \in \mathbb{R}^{n \times m}$$

for a given $x(0) = x_0 \in \mathbb{R}^n$

$$J(x_0; u(\cdot)) = \int_0^\infty x^{\mathrm{T}}(\tau) Q x(\tau) + u^{\mathrm{T}}(\tau) R u(\tau)\, d\tau$$

$$Q \in \mathbb{R}^{n \times n},\, Q = Q^{\mathrm{T}} \geq 0 \qquad R \in \mathbb{R}^{m \times m},\, R = R^{\mathrm{T}} > 0$$

optimal control problem

$$\inf_{\substack{u(\tau) \\ \tau \in [\,0, \infty\,)}} J(x_0; u(\cdot))$$

linear quadratic regulator problem

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x(0) = x_0 \qquad t \in [\,0, \infty\,)$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

$$A \in \mathbb{R}^{n \times n} \qquad B \in \mathbb{R}^{n \times m}$$

let $P = P^{\mathrm{T}} > 0$ be a solution to

$$0 = PA + A^{\mathrm{T}}P - PBR^{-1}B^{\mathrm{T}}P + Q$$

the optimal control is given by

$$u(t) = -\frac{1}{2}R^{-1}B^{\mathrm{T}}\frac{\partial V}{\partial x}(x) = -R^{-1}B^{\mathrm{T}}Px(t)$$

and the optimal cost is

$$\inf_{\substack{u(\tau)\\ \tau\in[\,0,\infty\,)}} J(x_0, u(\cdot)) = V(x_0) = x_0^{\mathrm{T}}Px_0$$

example

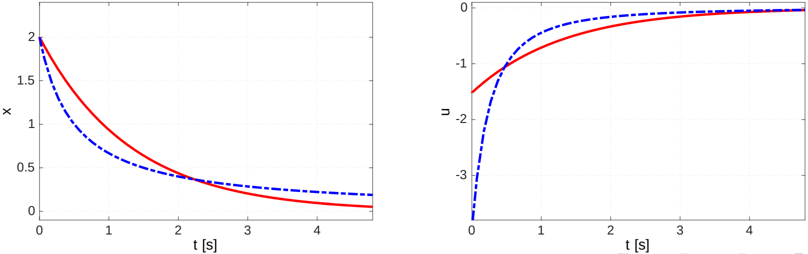
linear quadratic regulator problem

$$\dot{x}(t) = u(t) \qquad x(0) = x_0 \qquad t \in [\,0, \infty\,)$$

$$x(t) \in \mathbb{R} \qquad u(t) \in \mathbb{R}$$

$$J(x_0; u(\cdot)) = \int_0^\infty x^4(\tau) + u^4(\tau)\, d\tau \qquad x_0 = 2$$

optimal control $\quad u^* = -\left(\frac{1}{3}\right)^{1/4} x \qquad\qquad u = -x^2$



example

linear quadratic regulator problem

$$\dot{x}(t) = u(t) \qquad x(0) = x_0 \qquad t \in [\,0, \infty\,)$$

$$x(t) \in \mathbb{R} \qquad u(t) \in \mathbb{R}$$

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optimal control $\quad u^* = -\left(\frac{1}{3}\right)^{1/4} x$

example

linear quadratic regulator problem

$$\dot{x}(t) = 0x(t) + 1u(t) \qquad x(0) = x_0 \qquad t \in [\,0, \infty\,)$$

$$x(t) \in \mathbb{R} \qquad u(t) \in \mathbb{R}$$

$$J(x_0; u(\cdot)) = \int_0^\infty x^4(\tau) + u^4(\tau)\,d\tau$$

optimal control $\quad u^* = -\left(\frac{1}{3}\right)^{1/4} x$

example

linear quadratic regulator problem

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$$J(x_0; u(\cdot)) = \int_0^\infty x^2(\tau) + u^2(\tau)\,d\tau \qquad Q = 1 \quad R = 1$$

optimal control $\quad u^* = -\left(\frac{1}{3}\right)^{1/4} x$

example

linear quadratic regulator problem

$$\dot{x}(t) = 0x(t) + 1u(t) \qquad x(0) = x_0 \qquad t \in [\,0, \infty\,)$$

$$x(t) \in \mathbb{R} \qquad u(t) \in \mathbb{R}$$

$$J(x_0; u(\cdot)) = \int_0^{\infty} x^2(\tau) + u^2(\tau)\, d\tau \qquad Q = 1 \quad R = 1$$

optimal control $\quad u^* = -R^{-1} B^{\mathrm{T}} P x(t) = -x(t)$

example

linear quadratic regulator problem

$$\dot{x}(t) = 0x(t) + 1u(t) \qquad x(0) = x_0 \qquad t \in [\,0, \infty\,)$$

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optimal control $\quad u^* = -R^{-1}B^{\mathrm{T}}Px(t) = -x(t)$

$$0 = PA + A^{\mathrm{T}}P - PBR^{-1}B^{\mathrm{T}}P + Q$$

example

linear quadratic regulator problem

$$\dot{x}(t) = 0x(t) + 1u(t) \qquad x(0) = x_0 \qquad t \in [\,0, \infty\,)$$

$$x(t) \in \mathbb{R} \qquad u(t) \in \mathbb{R}$$

$$J(x_0; u(\cdot)) = \int_0^\infty x^2(\tau) + u^2(\tau)\, d\tau \qquad Q = 1 \quad R = 1$$

optimal control $\quad u^* = -R^{-1}B^{\mathrm{T}}Px(t) = -x(t)$

$$0 = PA + A^{\mathrm{T}}P - PBR^{-1}B^{\mathrm{T}}P + Q$$

$$0 = -P^2 + 1 \qquad P = 1, -1 \qquad P = 1 > 0$$

example

linear quadratic regulator problem

$$\dot{x}(t) = 0x(t) + 1u(t) \qquad x(0) = x_0 \qquad t \in [\,0, \infty\,)$$

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$$J(x_0; u(\cdot)) = \int_0^\infty x^2(\tau) + u^2(\tau)\, d\tau \qquad x_0 = 2$$

optimal control $u^* = -R^{-1}B^{\mathrm{T}}Px(t) = -x(t) \qquad u = -2x(t)$

example

linear quadratic regulator problem

$$\dot{x}(t) = 0x(t) + 1u(t) \qquad x(0) = x_0 \qquad t \in [\,0, \infty\,)$$

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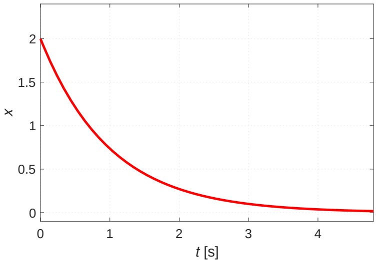
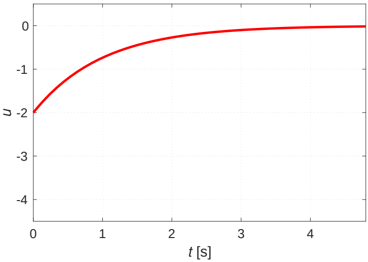




example

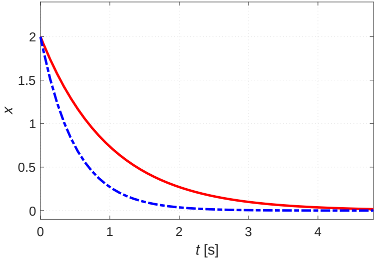
linear quadratic regulator problem

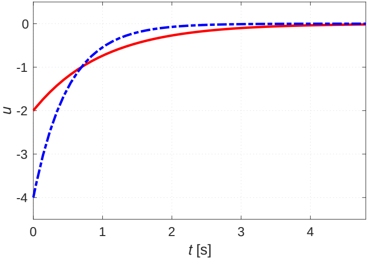
$$\dot{x}(t) = 0x(t) + 1u(t) \qquad x(0) = x_0 \qquad t \in [\,0, \infty\,)$$

$$x(t) \in \mathbb{R} \qquad u(t) \in \mathbb{R}$$

$$J(x_0; u(\cdot)) = \int_0^\infty x^2(\tau) + u^2(\tau)\,d\tau \qquad x_0 = 2$$

optimal control $\quad u^* = -R^{-1}B^{\mathrm{T}}Px(t) = -x(t) \qquad u = -2x(t)$





example

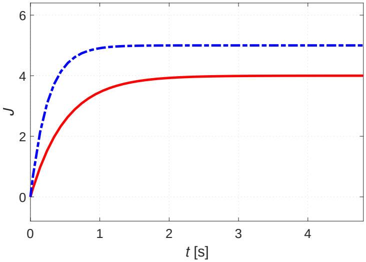
linear quadratic regulator problem

$$\dot{x}(t) = 0x(t) + 1u(t) \qquad x(0) = x_0 \qquad t \in [\,0, \infty\,)$$

$$x(t) \in \mathbb{R} \qquad u(t) \in \mathbb{R}$$

$$J(x_0; u(\cdot)) = \int_0^\infty x^2(\tau) + u^2(\tau)\,d\tau \qquad x_0 = 2$$

optimal control $\quad u^* = -R^{-1}B^{\mathrm{T}}Px(t) = -x(t) \qquad u = -2x(t)$



$$V(x_0) = x_0^{\mathrm{T}}Px_0 = 4$$

example 02

linear quadratic regulator problem

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$z(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

example 02

linear quadratic regulator problem

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$z(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

$$J(x_0; u(\cdot)) = \int_0^\infty z^2(\tau) + u^2(\tau)\, d\tau \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad R = 1$$

example 02

linear quadratic regulator problem

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$z(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

$$J(x_0; u(\cdot)) = \int_0^\infty z^2(\tau) + u^2(\tau) d\tau \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad R = 1$$

$$0 = PA + A^{\mathrm{T}} P - PBR^{-1} B^{\mathrm{T}} P + Q$$

example 02

linear quadratic regulator problem

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$z(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

$$J(x_0; u(\cdot)) = \int_0^\infty z^2(\tau) + u^2(\tau)\, d\tau \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad R = 1$$

$$0 = PA + A^{\mathrm{T}} P - PBR^{-1} B^{\mathrm{T}} P + Q$$

$$0 = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} 1 [0 \quad 1] \begin{bmatrix} p_{11} \\ p_{12} \end{bmatrix}$$

example 02

linear quadratic regulator problem

$$0 = \begin{bmatrix} 0 & p_{11} - p_{12} \\ p_{11} - p_{12} & 2(p_{12} - p_{22}) \end{bmatrix} - \begin{bmatrix} p_{12}^2 & p_{12}p_{22} \\ p_{12}p_{22} & p_{22}^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

example 02

linear quadratic regulator problem

$$0 = \begin{bmatrix} 0 & p_{11} - p_{12} \\ p_{11} - p_{12} & 2(p_{12} - p_{22}) \end{bmatrix} - \begin{bmatrix} p_{12}^2 & p_{12}p_{22} \\ p_{12}p_{22} & p_{22}^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$0 = -p_{12}^2 + 1$$

$$0 = p_{11} - p_{12} - p_{12}p_{22}$$

$$0 = 2p_{12} - 2p_{22} - p_{22}^2$$

example 02

linear quadratic regulator problem

$$0 = \begin{bmatrix} 0 & p_{11} - p_{12} \\ p_{11} - p_{12} & 2(p_{12} - p_{22}) \end{bmatrix} - \begin{bmatrix} p_{12}^2 & p_{12}p_{22} \\ p_{12}p_{22} & p_{22}^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$0 = -p_{12}^2 + 1$$

$$0 = p_{11} - p_{12} - p_{12}p_{22}$$

$$0 = 2p_{12} - 2p_{22} - p_{22}^2$$

$$P = \begin{bmatrix} \sqrt{3} & 1 \\ 1 & \sqrt{3} - 1 \end{bmatrix}$$

example 02

linear quadratic regulator problem

$$0 = \begin{bmatrix} 0 & p_{11} - p_{12} \\ p_{11} - p_{12} & 2(p_{12} - p_{22}) \end{bmatrix} - \begin{bmatrix} p_{12}^2 & p_{12}p_{22} \\ p_{12}p_{22} & p_{22}^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$0 = -p_{12}^2 + 1$$

$$0 = p_{11} - p_{12} - p_{12}p_{22}$$

$$0 = 2p_{12} - 2p_{22} - p_{22}^2$$

$$P = \begin{bmatrix} \sqrt{3} & 1 \\ 1 & \sqrt{3} - 1 \end{bmatrix}$$

$$\begin{aligned} u(t) = -R^{-1}B^{\mathrm{T}}Px(t) &= -1 [\, 0 \quad 1 \,] \begin{bmatrix} \sqrt{3} & 1 \\ 1 & \sqrt{3} - 1 \end{bmatrix} x(t) \\ &= -[\, 1 \quad \sqrt{3} - 1 \,] x(t) \end{aligned}$$

example 02

linear quadratic regulator problem

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$z(t) = [\, 1 \quad 0 \,] x(t)$$

$$J(x_0; u(\cdot)) = \int_0^\infty z^2(\tau) + u^2(\tau) d\tau \qquad x_0 = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

optimal control $\quad u^* = -[\, 1 \quad \sqrt{3} - 1 \,] x(t) \qquad u = -[\, 1 \quad 2 \,] x(t)$

example 02

linear quadratic regulator problem

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$z(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

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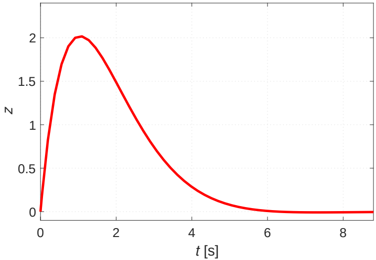

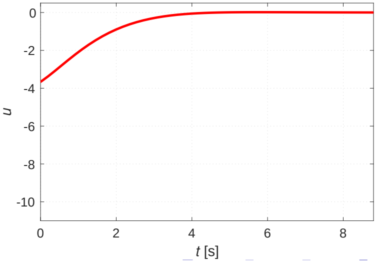

example 02

linear quadratic regulator problem

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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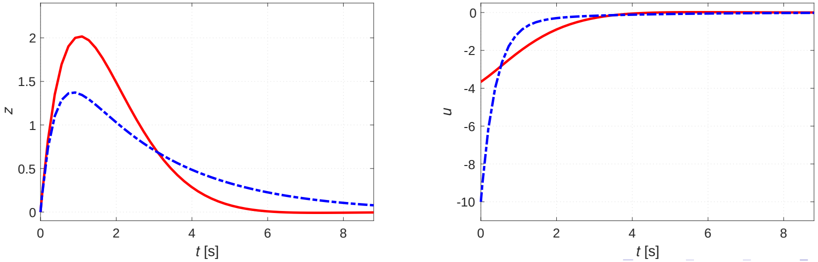

example 02

linear quadratic regulator problem

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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$$J(x_0; u(\cdot)) = \int_0^\infty z^2(\tau) + u^2(\tau) d\tau \qquad x_0 = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

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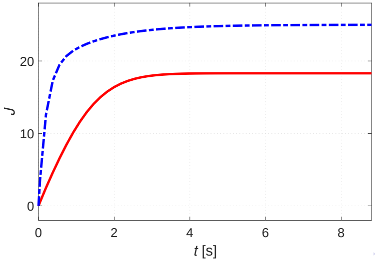
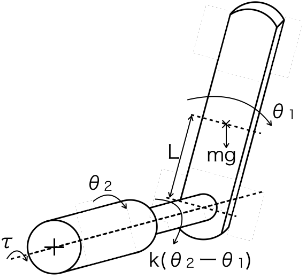


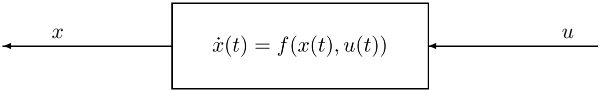
robotic manipulator



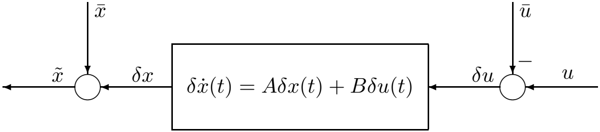
$$\frac{d}{dt}\begin{bmatrix}\theta_1\\ \theta_2\\ \dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix} = \begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ -\frac{k}{J_1}\theta_1 + \frac{k}{J_1}\theta_2 + \frac{mgL}{J_1}\sin\theta_1\\ \frac{k}{J_2}\theta_1 - \frac{k}{J_2}\theta_2 + \frac{1}{J_2}\tau\end{bmatrix}$$

nonlinear dynamical systems and linear approximations

$$\dot{x}(t) = f(x(t), u(t)) \qquad x(t) \in \mathbb{R}^n \quad u(t) \in \mathbb{R}^m$$



$$\delta\dot{x}(t) = A\delta x(t) + B\delta u(t) \qquad A \in \mathbb{R}^{n\times n} \quad B \in \mathbb{R}^{n\times m}$$

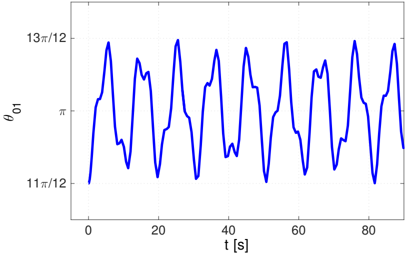


$$A = \left(\frac{\partial f}{\partial x}(\bar{x}, \bar{u})\right)^{\mathrm{T}} \qquad B = \left(\frac{\partial f}{\partial u}(\bar{x}, \bar{u})\right)^{\mathrm{T}}$$

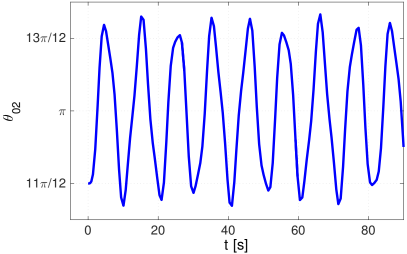
numerical simulation

robotic manipulator

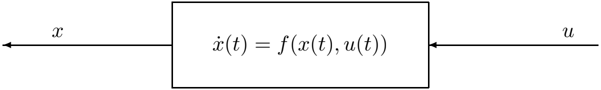
$$(\bar{x}, \bar{u}) = \left(\begin{bmatrix} \pi \\ \pi \\ 0 \\ 0 \end{bmatrix}, 0 \right) \quad x(0) = \begin{bmatrix} 11\pi/12 \\ 11\pi/12 \\ 0 \\ 0 \end{bmatrix} \quad \left(\delta x(0) = \begin{bmatrix} -\pi/12 \\ -\pi/12 \\ 0 \\ 0 \end{bmatrix} \right) \quad u(\cdot) = 0$$



θ_1



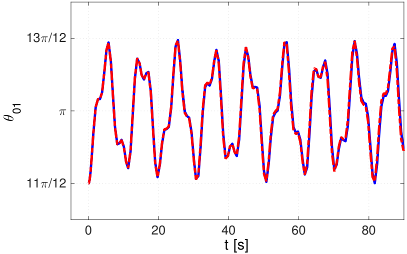
θ_2



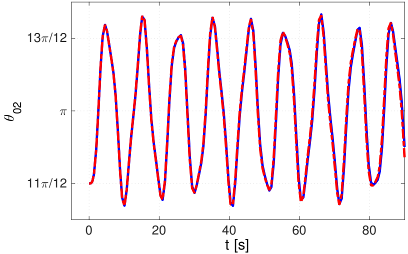
numerical simulation

robotic manipulator

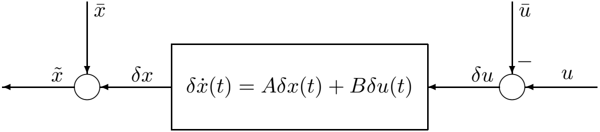
$$(\bar{x}, \bar{u}) = \left(\begin{bmatrix} \pi \\ \pi \\ 0 \\ 0 \end{bmatrix}, 0 \right) \quad x(0) = \begin{bmatrix} 11\pi/12 \\ 11\pi/12 \\ 0 \\ 0 \end{bmatrix} \quad \left(\delta x(0) = \begin{bmatrix} -\pi/12 \\ -\pi/12 \\ 0 \\ 0 \end{bmatrix} \right) \quad u(\cdot) = 0$$



θ_1



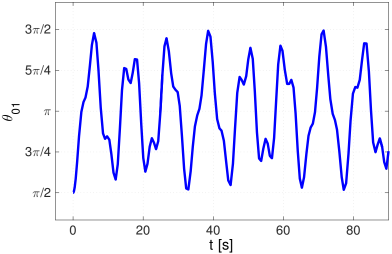
θ_2



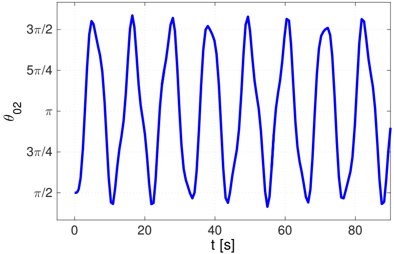
numerical simulation

robotic manipulator

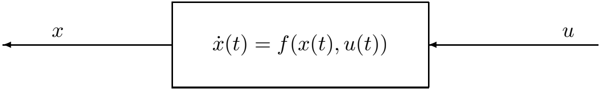
$$(\bar{x}, \bar{u}) = \left(\begin{bmatrix} \pi \\ \pi \\ 0 \\ 0 \end{bmatrix}, 0 \right) \quad x(0) = \begin{bmatrix} \pi/2 \\ \pi/2 \\ 0 \\ 0 \end{bmatrix} \quad \left(\delta x(0) = \begin{bmatrix} -\pi/2 \\ -\pi/2 \\ 0 \\ 0 \end{bmatrix} \right) \quad u(\cdot) = 0$$



θ_1



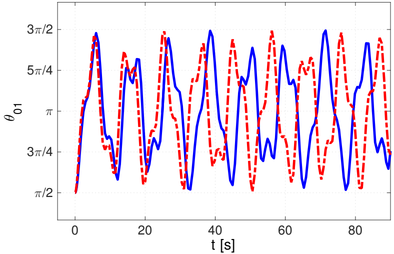
θ_2



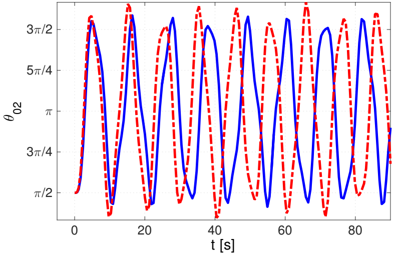
numerical simulation

robotic manipulator

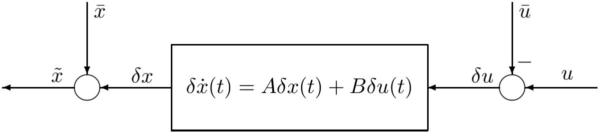
$$(\bar{x}, \bar{u}) = \left(\begin{bmatrix} \pi \\ \pi \\ 0 \\ 0 \end{bmatrix}, 0 \right) \quad x(0) = \begin{bmatrix} \pi/2 \\ \pi/2 \\ 0 \\ 0 \end{bmatrix} \quad \left(\delta x(0) = \begin{bmatrix} -\pi/2 \\ -\pi/2 \\ 0 \\ 0 \end{bmatrix} \right) \quad u(\cdot) = 0$$



θ_1



θ_2



linear approximation

robotic manipulator

$$(\bar{x}, \bar{u}) = \left(\begin{bmatrix} \bar{q} \\ \bar{q} - \frac{mgL}{k} \sin \bar{q} \\ 0 \\ 0 \end{bmatrix}, -mgL \sin \bar{q} \right) \qquad \text{for some } \bar{q} \in [\, \pi, -\pi \,)$$

linear approximation

robotic manipulator

$$(\bar{x}, \bar{u}) = \left(\begin{bmatrix} \bar{q} \\ \bar{q} - \frac{mgL}{k} \sin \bar{q} \\ 0 \\ 0 \end{bmatrix}, -mgL \sin \bar{q} \right) \qquad \text{for some } \bar{q} \in [\,\pi, -\pi\,)$$

$$\delta \dot{x}(t) = A \delta x + B \delta u$$

linear approximation
robotic manipulator

$$(\bar{x}, \bar{u}) = (\begin{bmatrix} \bar{q} \\ \bar{q} - \frac{mgL}{k}\sin\bar{q} \\ 0 \\ 0 \end{bmatrix}, -mgL\sin\bar{q}) \qquad \text{for some } \bar{q} \in [\,\pi, -\pi\,)$$

$$\delta\dot{x}(t) = A\delta x + B\delta u$$

$$A = \left(\frac{\partial f}{\partial x}(\bar{x}, \bar{u})\right)^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{J_1} + \frac{mgL}{J_1}\cos\bar{q} & \frac{k}{J_1} & 0 & 0 \\ \frac{k}{J_2} & -\frac{k}{J_2} & 0 & 0 \end{bmatrix}$$

$$B = \left(\frac{\partial f}{\partial u}(\bar{x}, \bar{u})\right)^{\mathrm{T}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J_2} \end{bmatrix} \qquad x = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

optimal control of linearized system

robotic manipulator

$$\delta\dot{x}(t) = A\delta x(t) + B\delta u(t) \qquad A \in \mathbb{R}^{4\times 4} \quad B \in \mathbb{R}^{4\times 1}$$

optimal control of linearized system

robotic manipulator

$$\delta\dot{x}(t) = A\delta x(t) + B\delta u(t) \qquad A \in \mathbb{R}^{4\times 4} \quad B \in \mathbb{R}^{4\times 1}$$

$$J(x_0; u(\cdot)) = \int_0^\infty \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \delta x(\tau) \right)^{\mathrm{T}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \delta x(\tau) + \delta u^2(\tau)\, d\tau$$

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R = 1$$

optimal control of linearized system

robotic manipulator

$$\delta\dot{x}(t) = A\delta x(t) + B\delta u(t) \qquad A \in \mathbb{R}^{4\times 4} \quad B \in \mathbb{R}^{4\times 1}$$

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$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R = 1$$

$$0 = PA + A^{\mathrm{T}}P - PBR^{-1}B^{\mathrm{T}}P + Q$$

optimal control of linearized system

robotic manipulator

$$\delta\dot{x}(t) = A\delta x(t) + B\delta u(t) \qquad A \in \mathbb{R}^{4\times 4} \quad B \in \mathbb{R}^{4\times 1}$$

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$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R = 1$$

$$0 = PA + A^{\mathrm{T}}P - PBR^{-1}B^{\mathrm{T}}P + Q \quad P = \begin{bmatrix} 18.2 & -3.3 & 1.3 & -1.4 \\ -3.3 & 11.4 & 5.9 & 3.8 \\ 1.3 & 5.9 & 11.1 & 1.6 \\ -1.4 & 3.8 & 1.6 & 2.7 \end{bmatrix}$$

optimal control of linearized system

robotic manipulator

$$\delta\dot{x}(t) = A\delta x(t) + B\delta u(t) \qquad A \in \mathbb{R}^{4\times 4} \quad B \in \mathbb{R}^{4\times 1}$$

$$J(x_0; u(\cdot)) = \int_0^\infty (\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \delta x(\tau))^{\mathrm{T}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \delta x(\tau) + \delta u^2(\tau) d\tau$$

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R = 1$$

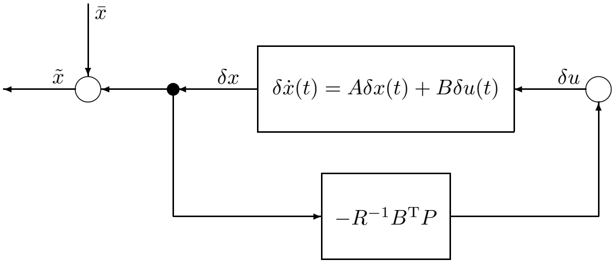
$$0 = PA + A^{\mathrm{T}}P - PBR^{-1}B^{\mathrm{T}}P + Q \quad P = \begin{bmatrix} 18.2 & -3.3 & 1.3 & -1.4 \\ -3.3 & 11.4 & 5.9 & 3.8 \\ 1.3 & 5.9 & 11.1 & 1.6 \\ -1.4 & 3.8 & 1.6 & 2.7 \end{bmatrix}$$

$$\delta u^* = -R^{-1}B^{\mathrm{T}}P = [\, 1.4 \quad -3.8 \quad -1.6 \quad -2.7 \,]$$

numerical simulation

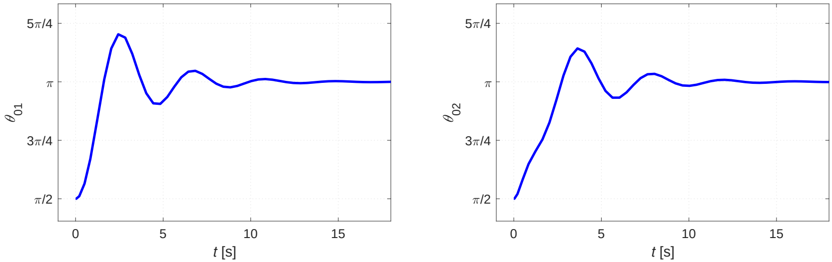
robotic manipulator

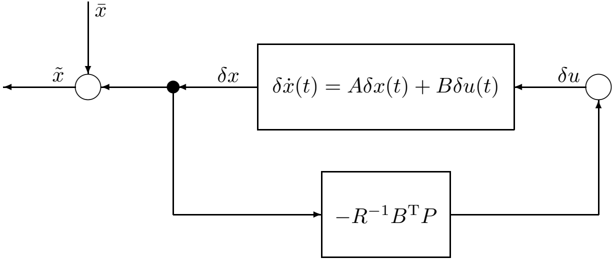
$$(\bar{x}, \bar{u}) = \left(\begin{bmatrix} \pi \\ \pi \\ 0 \\ 0 \end{bmatrix}, 0 \right) \quad x(0) = \begin{bmatrix} \pi/2 \\ \pi/2 \\ 0 \\ 0 \end{bmatrix} \quad \left(\delta x(0) = \begin{bmatrix} -\pi/2 \\ -\pi/2 \\ 0 \\ 0 \end{bmatrix} \right)$$



numerical simulation

robotic manipulator





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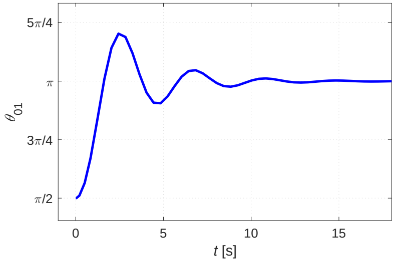

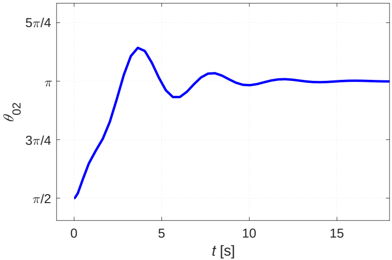

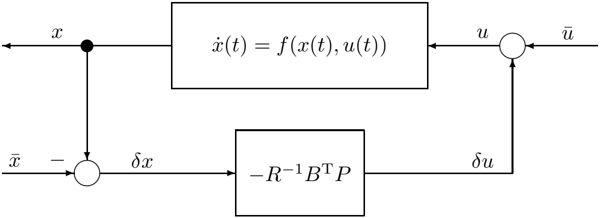

numerical simulation

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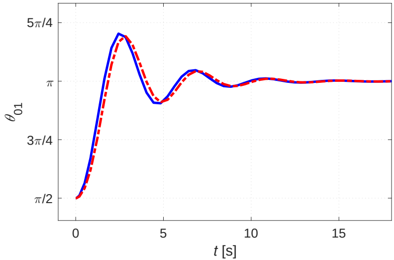
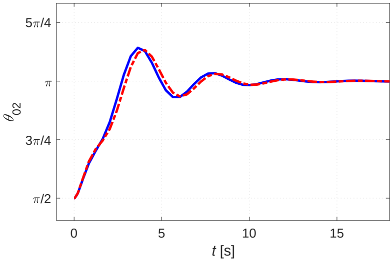
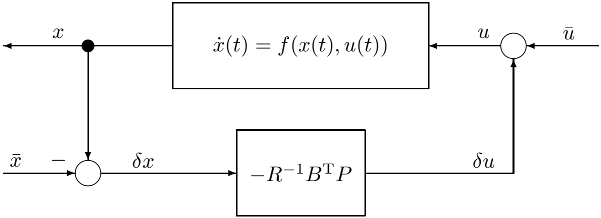
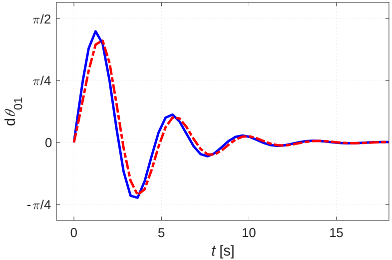


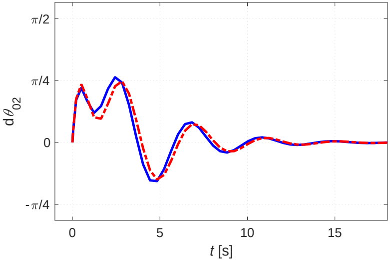


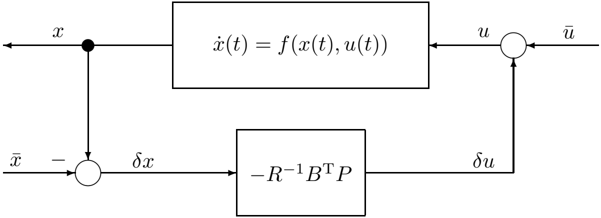


numerical simulation

robotic manipulator

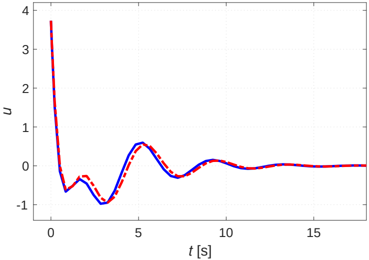


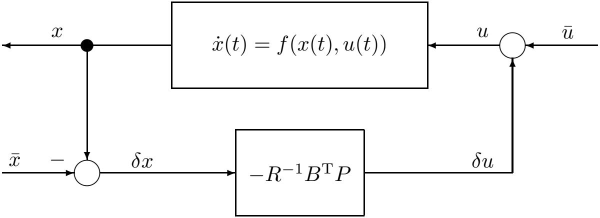




numerical simulation

robotic manipulator





contents

optimal control systems

- nonlinear dynamical systems and linear approximations
- dynamic programming
- the principle of optimality
- optimal control of finite state systems
- optimal control of discrete-time systems
- optimal control of continuous-time systems
- optimal control of linear systems
- decentralized optimal control
  - decentralization and integration via mechanism design

discrete-time linear quadratic regulator problem

$$x(t+1) = Ax(t) + Bu(t) \qquad x(0) = x_0 \qquad t \in \mathbb{Z}^+ = \{\, 0, 1, 2, \dots \,\}$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

$$A \in \mathbb{R}^{n \times n} \qquad B \in \mathbb{R}^{n \times m}$$

for a given $x(0) = x_0 \in \mathbb{R}^n$

$$J(x_0; u(\cdot)) = \sum_{\tau=0}^{\infty} (x^{\mathrm{T}}(\tau) Q x(\tau) + u^{\mathrm{T}}(\tau) R u(\tau))$$

$$Q \in \mathbb{R}^{n \times n}, Q = Q^{\mathrm{T}} \geq 0 \qquad R \in \mathbb{R}^{m \times m}, R = R^{\mathrm{T}} > 0$$

optimal control problem

$$\inf_{\substack{u(\tau) \\ \tau \in [\, 0, \infty \,)}} \quad J(x_0; u(\cdot))$$

discrete-time linear quadratic regulator problem

$$x(t+1) = Ax(t) + Bu(t) \qquad x(0) = x_0 \qquad t \in \mathbb{Z}^+ = \{\, 0, 1, 2, \ldots \,\}$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

$$A \in \mathbb{R}^{n \times n} \qquad B \in \mathbb{R}^{n \times m}$$

let $P = P^{\mathrm{T}} > 0$ be a solution to

$$P = A^{\mathrm{T}} P A - A^{\mathrm{T}} P B (B^{\mathrm{T}} P B + R)^{-1} B^{\mathrm{T}} P A + Q$$

the optimal control is given by

$$u(t) = -(B^{\mathrm{T}} P B + R)^{-1} B^{\mathrm{T}} P A x(t)$$

and the optimal cost is

$$\inf_{\substack{u(\tau) \\ \tau \in [\,0, \infty\,)}} J(x_0, u(\cdot)) = V(x_0) = x_0^{\mathrm{T}} P x_0$$

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