Advanced Control Systems Engineering I:

Optimal Control

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optimal control systems

- nonlinear dynamical systems and linear approximations
- dynamic programming
- the principle of optimality
- optimal control of finite state systems
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- decentralized optimal control
  - decentralization and integration via mechanism design

linear quadratic regulator problem

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x(0) = x_0 \qquad t \in [\,0, \infty\,)$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

$$A \in \mathbb{R}^{n \times n} \qquad B \in \mathbb{R}^{n \times m}$$

for a given $x(0) = x_0 \in \mathbb{R}^n$

$$J(x_0; u(\cdot)) = \int_0^\infty x^{\mathrm{T}}(\tau) Q x(\tau) + u^{\mathrm{T}}(\tau) R u(\tau)\, d\tau$$

$$Q \in \mathbb{R}^{n \times n},\, Q = Q^{\mathrm{T}} \geq 0 \qquad R \in \mathbb{R}^{m \times m},\, R = R^{\mathrm{T}} > 0$$

optimal control problem

$$\inf_{\substack{u(\tau) \\ \tau \in [\,0,\infty\,)}} \; J(x_0; u(\cdot))$$

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let $P = P^{\mathrm{T}} > 0$ be a solution to

$$0 = PA + A^{\mathrm{T}}P - PBR^{-1}B^{\mathrm{T}}P + Q$$

the optimal control is given by

$$u(t) = -\frac{1}{2}R^{-1}B^{\mathrm{T}}\frac{\partial V}{\partial x}(x) = -R^{-1}B^{\mathrm{T}}Px(t)$$

and the optimal cost is

$$\inf_{\substack{u(\tau) \\ \tau \in [\,0, \infty\,)}} J(x_0, u(\cdot)) = V(x_0) = x_0^{\mathrm{T}}Px_0$$

discrete-time linear quadratic regulator problem

$$x(t+1) = Ax(t) + Bu(t) \qquad x(0) = x_0 \qquad t \in \mathbb{Z}^+ = \{\,0, 1, 2, \ldots\,\}$$

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$$\inf_{\substack{u(\tau) \\ \tau\in[\,0,\infty\,)}} J(x_0; u(\cdot))$$

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$$u(t) = -(B^{\mathrm{T}}PB + R)^{-1}B^{\mathrm{T}}PAx(t)$$

and the optimal cost is

$$\inf_{\substack{u(\tau) \\ \tau \in [\,0, \infty\,)}} \quad J(x_0, u(\cdot)) = V(x_0) = x_0^{\mathrm{T}} P x_0$$

exercise 1

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x(0) = x_0 \qquad t \in [\,0, \infty\,)$$

$$J(x_0; u(\cdot)) = \int_0^\infty x^{\mathrm{T}}(\tau)Qx(\tau) + u^{\mathrm{T}}(\tau)Ru(\tau)\,d\tau$$

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$$\dot{x}(t) = 2x(t) + u(t)$$

$$J(x_0; u(\cdot)) = \int_0^\infty 5x^2(\tau) + u^2(\tau) d\tau$$

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$$u(t) = -R^{-1}B^{\mathrm{T}}Px(t)$$

$$\dot{x}(t) = 2x(t) + u(t) = 2x(t) + 1u(t)$$

$$J(x_0; u(\cdot)) = \int_0^\infty 5x^2(\tau) + u^2(\tau)\,d\tau = \int_0^\infty x(\tau) \times 5 \times x(\tau) + u(\tau) \times 1 \times u(\tau)\,d\tau$$

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$$\dot{x}(t) = 2x(t) + 1u(t)$$

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$$\begin{aligned} 0 &= 2p + 2p - p \times 1 \times 1 \times 1 \times p + 5 \\ &= -(p-5)(p+1) \qquad p = 5 > 0 \end{aligned}$$

$$u(t) = -1 \times 1 \times 5x(t) = -5x(t)$$

exercise 2

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$J(x_0; u(\cdot)) = \int_0^\infty 4x_1^2(\tau) + 4x_2(\tau)^2 + u^2(\tau)\, d\tau \qquad Q = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \qquad R = 1$$

$$0 = PA + A^{\mathrm{T}}P - PBR^{-1}B^{\mathrm{T}}P + Q \qquad P = P^{\mathrm{T}} > 0$$

$$u(t) = -R^{-1}B^{\mathrm{T}}Px(t)$$

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$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad Q = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \quad R = 1$$

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$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad Q = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \quad R = 1$$

$$0 = PA + A^{\mathrm{T}}P - PBR^{-1}B^{\mathrm{T}}P + Q \qquad P = P^{\mathrm{T}} > 0$$

$$\begin{aligned} PA + A^{\mathrm{T}}P &= \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \\ &= \begin{bmatrix} 0 & p_{11} + p_{12} \\ p_{11} + p_{12} & 2(p_{12} + p_{22}) \end{bmatrix} \end{aligned}$$

$$\begin{aligned} PBR^{-1}B^{\mathrm{T}}P &= P \begin{bmatrix} 0 \\ 1 \end{bmatrix} 1 [\, 0 \quad 1 \,] P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \\ &= \begin{bmatrix} p_{12}^2 & p_{12}p_{22} \\ p_{12}p_{22} & p_{22}^2 \end{bmatrix} \end{aligned}$$

exercise 2

$$0 = \begin{bmatrix} 0 & p_{11} + p_{12} \\ p_{11} + p_{12} & 2(p_{12} + p_{22}) \end{bmatrix} - \begin{bmatrix} p_{12}^2 & p_{12}p_{22} \\ p_{12}p_{22} & p_{22}^2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$0 = -p_{12}^2 + 4 \qquad \begin{aligned} 0 &= p_{11} + p_{12} - p_{12}p_{22} \\ 0 &= 2p_{12} + 2p_{22} - p_{22}^2 + 4 \end{aligned}$$

$$\begin{aligned} p_{12}^2 &= 4 \\ (p_{22} - 4)(p_{22} + 2) & \end{aligned} \qquad \begin{aligned} p_{12} &= 2 \\ p_{22} &= 4 \\ p_{11} &= 6 \end{aligned} \qquad P = \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} > 0$$

$$u(t) = -R^{-1}B^{\mathrm{T}}Px(t) = -1\,[\,0 \quad 1\,] \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} = -[\,2 \quad 4\,]x(t)$$

exercise 3

$$x(t+1) = Ax(t) + Bu(t) \qquad x(0) = x_0 \qquad t \in \mathbb{Z}^+ = \{\, 0, 1, 2, \ldots \,\}$$

$$J(x_0; u(\cdot)) = \sum_{\tau=0}^{\infty} (x^{\mathrm{T}}(\tau) Q x(\tau) + u^{\mathrm{T}}(\tau) R u(\tau))$$

$$P = A^{\mathrm{T}} P A - A^{\mathrm{T}} P B (B^{\mathrm{T}} P B + R)^{-1} B^{\mathrm{T}} P A + Q \qquad P = P^{T}$$

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$$x(t+1) = x(t) + 2u(t)$$

$$J(x_0; u(\cdot)) = \sum_{\tau=0}^{\infty} (4x^2(\tau) + u^2(\tau))$$

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$$u(t) = -(B^{\mathrm{T}}PB + R)^{-1}B^{\mathrm{T}}PAx(t)$$

$$p = 1p1 - 1p2(2p2+1)^{-1}2p1 + 4$$

$$0 = 4(p^2 - 4p - 1) \qquad p = 2 + \sqrt{5} > 0$$

$$u(t) = -(2(2+\sqrt{5})2 + 1)^{-1}2(2+\sqrt{5})1x(t) = -\frac{2(2+\sqrt{5})}{9+4\sqrt{5}}x(t)$$

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