$$
\begin{aligned}
x(t) &\in X = \{x_1, x_2, \dots, x_n\} \\
u(t) &\in U = \{u_1, u_2, \dots, u_m\} \\
t &\in T = \{t_0, t_0+1, \dots, t_\mathrm{f}\}
\end{aligned}
$$

$$
\begin{aligned}
\phi &: X \times U \to X & \qquad & x(t+1) = \phi(x(t), u(t)) \\
\ell &: X \times U \to \mathbb{R} & & \ell(x(t), u(t)) \\
\ell_\mathrm{f} &: X \to \mathbb{R} & & \ell_\mathrm{f}(x(t_\mathrm{f}))
\end{aligned}
$$

Let $x(t_0) = x_0 \in X$ be given, and consider the optimal control problem:

$$
J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))
$$

$$
\inf_{\substack{u(\tau)\in U \\ \tau \in T}} J(t_0, x_0; u(\cdot))
$$

Define the cost-to-go:

$$
V : T \times X \to \mathbb{R} \qquad\qquad V(t, x) = \inf_{\substack{u(\tau)\in U \\ \tau \in \{t, t+1, \dots, t_\mathrm{f}\}}} J(t, x; u(\cdot))
$$

Bellman equation:

$$
\begin{aligned}
V(t_\mathrm{f}, x) &= \ell_\mathrm{f}(x) & \qquad & \text{for all } x \in X \\
V(t, x) &= \inf_{u \in U} \{\ell(x, u) + V(t+1, \phi(x, u))\} & & \text{for all } x \in X \text{ and all } t \in \{t_0, t_{0+1}, \dots, t_\mathrm{f} - 1\}
\end{aligned}
$$

State feedback control:

$$
u(t) = u(x(t)) = \arg\min_{u \in U} \{ \underbrace{\ell(x(t), u) + V(t+1, \phi(x(t), u))}_{\text{computed using the measured state } x(t)} \}
$$

$$
x(t+1) = \phi(x(t), u(t)) \qquad x(t_0) = x_0 \in X
$$

example:

$$\begin{aligned} x(t) &\in X = \{x_1, x_2, x_4, x_5\} \\ u(t) &\in U = \{u_1, u_2, u_3\} \\ t &\in T = \{1, 2, 3\} \end{aligned}$$

$$\begin{aligned} \phi &: X \times U \to X & \qquad & x(t+1) = \phi(x(t), u(t)) \\ \ell &: X \times U \to \mathbb{R} & & \ell(x(t), u(t)) \\ \ell_{\mathrm{f}} &: X \to \mathbb{R} & & \ell_{\mathrm{f}}(x(t_{\mathrm{f}})) \end{aligned}$$

ϕ, ℓ and ℓ_{f} are given as follows:

$\phi(x_i, u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i, u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$\ell_{\mathrm{f}}(x_i)$
x_1	1
x_2	2
x_3	3
x_4	4
x_5	5

Let $x(1) \in X$ be given, and consider the optimal control problem:

$$\begin{aligned} J(1, x(1); u(\cdot)) &= \sum_{\tau=1}^{2} \ell(x(\tau), u(\tau)) + \ell_{\mathrm{f}}(x(3)) \\ &= \ell(x(1), u(1)) + \ell(x(2), u(2)) + \ell_{\mathrm{f}}(x(3)) \end{aligned}$$

$$\inf_{u(1), u(2) \in U} J(1, x(1); u(\cdot))$$

Bellman equation:

$$\begin{aligned} t = 3: &\qquad V(3, x) = \ell_{\mathrm{f}}(x) && \text{for all } x \in X \\ t = 2: &\qquad V(2, x) = \inf_{u \in U} \{\ell(x, u) + V(3, \phi(x, u))\} && \text{for all } x \in X \\ t = 1: &\qquad V(1, x) = \inf_{u \in U} \{\ell(x, u) + V(2, \phi(x, u))\} && \text{for all } x \in X \end{aligned}$$

for $t = t_\mathrm{f} = 3$:

$$V(3, x) = \ell_\mathrm{f}(x) \qquad \text{for all } x \in X$$

	$\ell_\mathrm{f}(x_i)$
x_1	1
x_2	2
x_3	3
x_4	4
x_5	5

	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1			1
x_2			2
x_3			3
x_4			4
x_5			5

for $t = t_\mathrm{f} - 1 = 2$:

$$V(2,x) = \inf_{u \in U} \{\ell(x,u) + V(3,\phi(x,u))\} \qquad \text{for all } x \in X$$

$$V(2,x_1) = \inf_{u \in U} \{\underbrace{\ell(x_1,u)}_{\substack{1\\5\\3}} + \underbrace{V(3,\phi(x_1,u))}_{\substack{3\\1\\5}}\} \qquad = 4$$

$$V(2,x_2) = \inf_{u \in U} \{\underbrace{\ell(x_2,u)}_{\substack{4\\1\\2}} + \underbrace{V(3,\phi(x_2,u))}_{\substack{4\\3\\2}}\} \qquad = 4$$

$$V(2,x_3) = \inf_{u \in U} \{\underbrace{\ell(x_3,u)}_{\substack{2\\3\\1}} + \underbrace{V(3,\phi(x_3,u))}_{\substack{2\\5\\4}}\} \qquad = 4$$

$$V(2,x_4) = \inf_{u \in U} \{\underbrace{\ell(x_4,u)}_{\substack{3\\4\\5}} + \underbrace{V(3,\phi(x_4,u))}_{\substack{1\\2\\1}}\} \qquad = 4$$

$$V(2,x_5) = \inf_{u \in U} \{\underbrace{\ell(x_5,u)}_{\substack{5\\2\\4}} + \underbrace{V(3,\phi(x_5,u))}_{\substack{5\\4\\3}}\} \qquad = 6$$

$\phi(x_i,u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i,u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$V(1,x_i)$	$V(2,x_i)$	$V(3,x_i)$
x_1		4	1
x_2		4	2
x_3		4	3
x_4		4	4
x_5		6	5

for $t = 1$:

$$V(1,x) = \inf_{u \in U} \{\ell(x,u) + V(2,\phi(x,u))\} \qquad \text{for all } x \in X$$

$$V(1,x_1) = \inf_{u \in U} \{\underbrace{\ell(x_1,u)}_{\substack{1\\5\\3}} + \underbrace{V(2,\phi(x_1,u))}_{\substack{4\\4\\6}}\} = 5$$

$$V(1,x_2) = \inf_{u \in U} \{\underbrace{\ell(x_2,u)}_{\substack{4\\1\\2}} + \underbrace{V(2,\phi(x_2,u))}_{\substack{4\\4\\4}}\} = 5$$

$$V(1,x_3) = \inf_{u \in U} \{\underbrace{\ell(x_3,u)}_{\substack{2\\3\\1}} + \underbrace{V(2,\phi(x_3,u))}_{\substack{4\\6\\4}}\} = 5$$

$$V(1,x_4) = \inf_{u \in U} \{\underbrace{\ell(x_4,u)}_{\substack{3\\4\\5}} + \underbrace{V(2,\phi(x_4,u))}_{\substack{4\\4\\4}}\} = 7$$

$$V(1,x_5) = \inf_{u \in U} \{\underbrace{\ell(x_5,u)}_{\substack{5\\2\\4}} + \underbrace{V(2,\phi(x_5,u))}_{\substack{6\\4\\3}}\} = 6$$

$\phi(x_i,u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i,u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$V(1,x_i)$	$V(2,x_i)$	$V(3,x_i)$
x_1	5	4	1
x_2	5	4	2
x_3	5	4	3
x_4	7	4	4
x_5	6	6	5

Let us consider the initial condition $x(1) = x_5$, then the optimal trajectory is given as follows:

$$
\begin{aligned}
x(1) &= x_5 \\
u(1) &= \arg\min_{u\in U}\{\underbrace{\ell(x_5,u)}_{\substack{5\\2\\4}} + \underbrace{V(2,\phi(x_5,u))}_{\substack{6\\4\\4}}\} && = u_2 \\
x(2) &= \phi(x_5,u_2) = x_4 \\
u(2) &= \arg\min_{u\in U}\{\underbrace{\ell(x_4,u)}_{\substack{3\\4\\5}} + \underbrace{V(3,\phi(x_4,u))}_{\substack{1\\2\\1}}\} && = u_1 \\
x(3) &= \phi(x_4,u_1) = x_1
\end{aligned}
$$

$$
\begin{aligned}
\min_{u(1),u(2)\in U} J(1,x_5;u(\cdot)) &= \ell(x_5,u_2) + \ell(x_4,u_1) + \ell_{\mathrm{f}}(x_1) \\
&= 2 + 3 + 1 = 6 = V(1,x_5)
\end{aligned}
$$

$\phi(x_i,u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i,u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$\ell_{\mathrm{f}}(x_i)$
x_1	1
x_2	2
x_3	3
x_4	4
x_5	5

	$V(1,x_i)$	$V(2,x_i)$	$V(3,x_i)$
x_1	5	4	1
x_2	5	4	2
x_3	5	4	3
x_4	7	4	4
x_5	6	6	5