$$x(t+1) = f(x(t), u(t)) \qquad x(t_0) = x_0 \qquad t \in T = \{t_0, t_0+1, \ldots, t_\mathrm{f}\}$$
$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

$$\begin{aligned} f &: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n & \qquad & x(t+1) = f(x(t), u(t)) \\ \ell &: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} & & \ell(x(t), u(t)) \\ \ell_\mathrm{f} &: \mathbb{R}^n \to \mathbb{R} & & \ell_\mathrm{f}(x(t_\mathrm{f})) \end{aligned}$$

Let $x(t_0) = x_0 \in \mathbb{R}^n$ be given, and consider the optimal control problem:

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

$$\inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau \in T}} J(t_0, x_0; u(\cdot))$$

Define the cost-to-go:

$$V : T \times \mathbb{R}^n \to \mathbb{R} \qquad\qquad V(t, x) = \inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau \in \{t, t+1, \ldots, t_\mathrm{f}\}}} J(t, x; u(\cdot))$$

Note that computing the cost-to-go $V(t_0, x_0)$ from the initial state x_0 at the initial time t_0 essentially amounts to minimize the cost $J(t_0, x_0; u(\cdot))$.

If $t = t_{\mathrm{f}}$:

$$\begin{aligned} V(t_{\mathrm{f}}, x) &= \inf_{u(t_{\mathrm{f}}) \in \mathbb{R}^m} J(t_{\mathrm{f}}, x; u(t_{\mathrm{f}})) \\ &= \inf_{u(t_{\mathrm{f}}) \in \mathbb{R}^m} \underbrace{\ell_{\mathrm{f}}(x)}_{\substack{\text{independent} \\ \text{of } u(t_{\mathrm{f}})}} = \ell_{\mathrm{f}}(x) \end{aligned}$$

If $t = t_{\mathrm{f}} - 1$:

$$\begin{aligned} V(t_{\mathrm{f}} - 1, x) &= \inf_{u(t_{\mathrm{f}}-1), u(t_{\mathrm{f}}) \in \mathbb{R}^m} J(t_{\mathrm{f}} - 1, x; u(\cdot)) \\ &= \inf_{u(t_{\mathrm{f}}-1), u(t_{\mathrm{f}}) \in \mathbb{R}^m} \{ \underbrace{\ell(x(t_{\mathrm{f}} - 1), u(t_{\mathrm{f}} - 1))}_{\text{independent of } u(t_{\mathrm{f}})} + \underbrace{\ell_{\mathrm{f}}(x(t_{\mathrm{f}}))}_{\substack{\text{depend on both} \\ u(t_{\mathrm{f}}-1) \text{ and } u(t_{\mathrm{f}})}} \} \\ &= \inf_{u(t_{\mathrm{f}}-1) \in \mathbb{R}^m} \{ \ell(x, u(t_{\mathrm{f}} - 1)) + \underbrace{\inf_{u(t_{\mathrm{f}}) \in \mathbb{R}^m} \ell_{\mathrm{f}}(x(t_{\mathrm{f}}))}_{= V(t_{\mathrm{f}}, x(t_{\mathrm{f}})) = V(t_{\mathrm{f}}, f(x, u(t_{\mathrm{f}} - 1)))} \} \\ &= \inf_{u(t_{\mathrm{f}}-1) \in \mathbb{R}^m} \{ \ell(x, u(t_{\mathrm{f}} - 1)) + V(t_{\mathrm{f}}, f(x, u(t_f - 1))) \} \\ &= \inf_{u \in \mathbb{R}^m} \{ \ell(x, u) + V(t_{\mathrm{f}}, f(x, u)) \} \end{aligned}$$

For $t < t_{\mathrm{f}}$:

$$\begin{aligned} V(t, x) &= \inf_{\substack{u(\tau) \in \mathbb{R}^m \\ \tau \in \{t, t+1, \ldots, t_{\mathrm{f}}\}}} J(t, x; u(\cdot)) \\ &= \inf_{\substack{u(\tau) \in \mathbb{R}^m \\ \tau \in \{t, t+1, \ldots, t_{\mathrm{f}}\}}} \{ \sum_{\tau = t}^{t_{\mathrm{f}} - 1} \ell(x(\tau), u(\tau)) + \ell_{\mathrm{f}}(x(t_{\mathrm{f}})) \} \\ &= \inf_{\substack{u(\tau) \in \mathbb{R}^m \\ \tau \in \{t, t+1, \ldots, t_{\mathrm{f}}\}}} \{ \underbrace{\ell(x, u(t))}_{\substack{\text{independent of } u(\tau), \\ \tau \in \{t+1, t+2, \ldots, t_{\mathrm{f}}\}}} + \underbrace{\sum_{\tau = t+1}^{t_{\mathrm{f}} - 1} \ell(x(\tau), u(\tau)) + \ell_{\mathrm{f}}(x(t_{\mathrm{f}}))}_{\text{depend on all } u(\tau),\ \tau \in \{t, t+1, \ldots, t_{\mathrm{f}}\}} \} \\ &= \inf_{u(t) \in \mathbb{R}^m} \{ \ell(x, u(t)) + \underbrace{\inf_{\substack{u(\tau) \in \mathbb{R}^m \\ \tau \in \{t+1, t+2, \ldots, t_{\mathrm{f}}\}}} \{ \sum_{\tau = t+1}^{t_{\mathrm{f}} - 1} \ell(x(\tau), u(\tau)) + \ell_{\mathrm{f}}(x(t_{\mathrm{f}})) \}}_{= V(t+1, x(t+1)) = V(t+1, f(x, u(t)))} \} \\ &= \inf_{u(t) \in \mathbb{R}^m} \{ \ell(x, u(t)) + V(t+1, f(x, u(t))) \} \\ &= \inf_{u \in \mathbb{R}^m} \{ \ell(x, u) + V(t+1, f(x, u)) \} \end{aligned}$$

Bellman equation:

$$\begin{aligned} V(t_{\mathrm{f}}, x) &= \ell_{\mathrm{f}}(x) && \text{for all } x \in \mathbb{R}^n \\ V(t, x) &= \inf_{u \in \mathbb{R}^m} \{\ell(x, u) + V(t+1, f(x, u))\} && \text{for all } x \in \mathbb{R}^n \text{ and all } t \in \{t_0, t_{0+1}, \ldots, t_{\mathrm{f}} - 1\} \end{aligned}$$

Let us suppose that the cost-to-go V has been determined. For a given state x at time t, the optimal input $u(t)$ is given as

$$u(t) = \arg\min_{u \in \mathbb{R}^m} \{\ell(x, u) + V(t+1, f(x, u))\}$$

This inspires the implantation of the optimal control in a state feed back form:

$$\begin{aligned} u(t) = u(x(t)) &= \arg\min_{u \in \mathbb{R}^m} \{ \underbrace{\ell(x(t), u) + V(t+1, f(x(t), u))}_{\text{computed using the measured state } x(t)} \} \\ x(t+1) &= f(x(t), u(t)) \qquad x(t_0) = x_0 \end{aligned}$$