Advanced Control Systems Engineering I:

Optimal Control

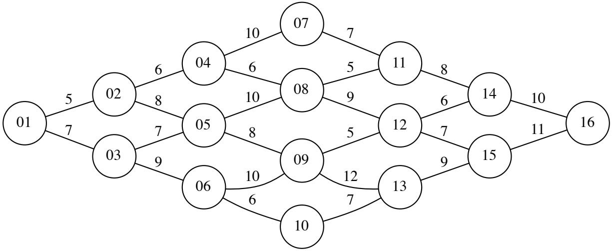
contents

optimal control

- nonlinear dynamical systems and linear approximations
- dynamic programming
- the principle of optimality
- optimal control of finite state systems
- optimal control of discrete-time systems
- optimal control of continuous-time systems
- optimal control of linear systems
- decentralized optimal control
  - decentralization and integration via mechanism design

finite state systems

optimal control problem



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$$t \in [\,0, 1, 2, 3, 4, 5, 6\,]$$

$$\inf_{u(\cdot)} J(0, x_1; u(\cdot)) = \inf_{u(\cdot)} \sum_{\tau=0}^{6} \ell(x(\tau), u(\tau))$$

minimum-cost path problem

- multistage decision process

optimal control problem

$$\dot{x}(t) = f(x(t), u(t)) \qquad x(t_0) = x_0 \qquad t \in [\, t_0, t_\mathrm{f} \,]$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

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dynamic programming

DYNAMIC

PROGRAMMING

BY

RICHARD BELLMAN

In his 1957 book, R. E. Bellman wrote:

dynamic programming

DYNAMIC

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> In order to answer this, let us turn to the previously enunciated principle that it is the *structure* of the policy which is essential. What does this mean precisely? It means that we wish to know the characteristics of the system which determine the decision to be made at any particular stage of the process. Put another way, in place of determining the optimal sequence of decisions from some *fixed* state of the system, we wish to determine the optimal decision to be made at *any* state of the system. Only if we know the latter, do we understand the intrinsic structure of the solution.
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principle of optimality

[Bellman, 1957, p. 83]

principle of optimality

§ 3. The principle of optimality

In each process, the functional equation governing the process was obtained by an application of the following intuitive:

PRINCIPLE OF OPTIMALITY. *An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.*

The mathematical transliteration of this simple principle will yield all the functional equations we shall encounter throughout the remainder of the book. A proof by contradiction is immediate.

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富山に居る先生が，出張で名古屋へ行く．時間最短で検索したところ，富山から東京まで新幹線で，東京から名古屋まで新幹線で，が最短経路だった．このとき，東京に居る先生が名古屋へ出張する際の時間最短経路は，東京から名古屋まで新幹線で，である．

principle of optimality

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$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

$$\inf_{u(\cdot)} J(t_0, x_0, u) = \inf_{u(\cdot)} \int_{t_0}^{t_\mathrm{f}} \ell(x(t), u(t), t)\, dt$$

The Principle of Optimality: Let $u^*(\cdot)$ be an optimal control that generates the trajectory $x(t)$, $t \in [\, t_0, t_\mathrm{f} \,]$, with $x(t_0) = x_0$. Then the trajectory $x(\cdot)$ from (t_0, x_0) to $(t_\mathrm{f}, x(t_\mathrm{f}))$ is optimal if and only if for all t_1, $t_2 \in [\, t_0, t_\mathrm{f} \,]$, the portion of the trajectory $x(\cdot)$ going from $(t_1, x(t_1))$ to $(t_2, x(t_2))$ optimizes the same cost functional over $[t_1, t_2]$, where $x(t_1) = x_1$ is a point on the optimal trajectory generated by $u^*(\cdot)$.

finite state systems

optimal control problem

$$x(t) \in X = \{x_1, x_2, \ldots, x_n\} \qquad u(t) \in U = \{u_1, u_2, \ldots, u_m\}$$

$$t \in \{t_0, t_0 + 1, \ldots, t_\mathrm{f}\}$$

finite state systems

optimal control problem

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finite state systems

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$\phi(x_i, u_j)$	u_1	u_2	$\cdots$	u_m
x_1	x_3	x_{n-2}	$\cdots$	x_1
x_2	x_2	x_8	$\cdots$	x_n
$\vdots$				
x_n	x_5	x_{n-7}	$\cdots$	x_2

finite state systems

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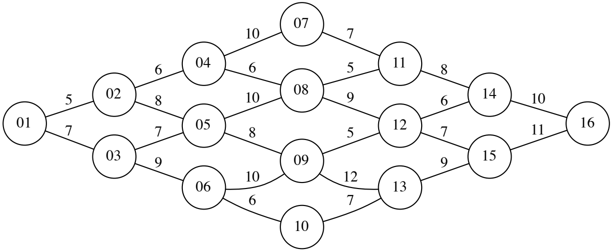
x_1	3	2	$\cdots$	-1
x_2	2	-2	$\cdots$	6
$\vdots$				
x_n	-1	5	$\cdots$	1.2

	$\ell_\mathrm{f}(x_i)$
x_1	3
x_2	2
$\vdots$	
x_n	-1

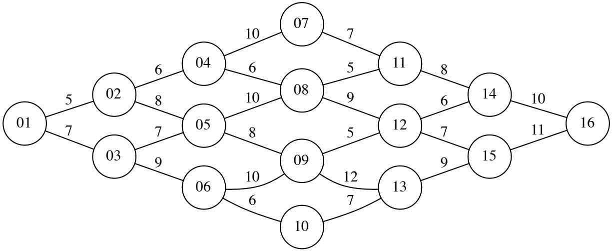
finite state systems

optimal control problem



finite state systems

optimal control problem



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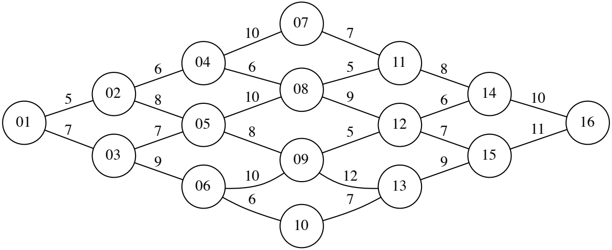
finite state systems

optimal control problem



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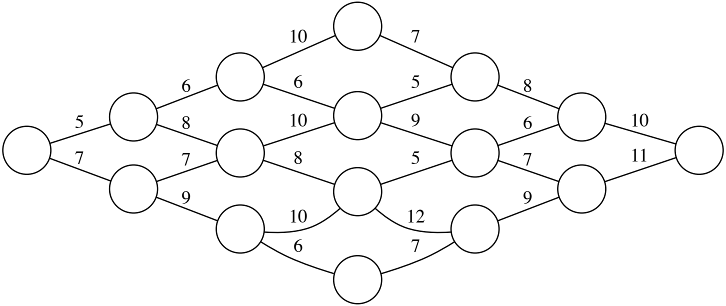
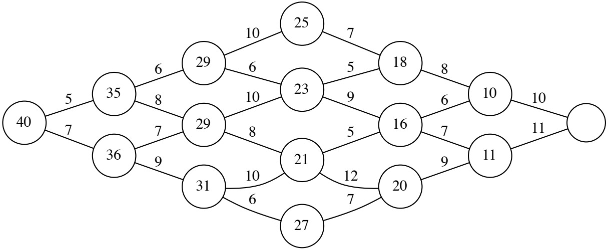
finite state systems

optimal control problem



finite state systems

optimal control problem



finite state systems

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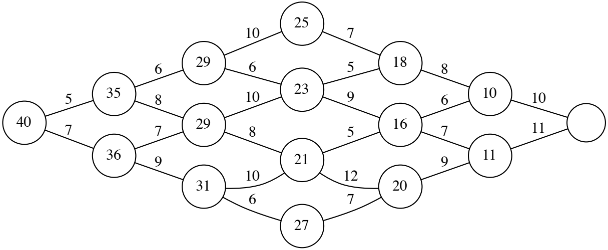
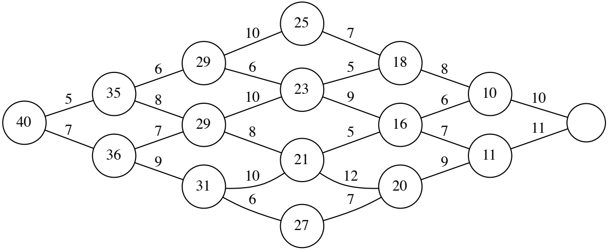


(naive) computational complexities

finite state systems

optimal control problem

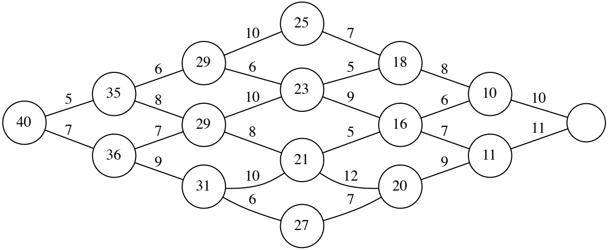


(naive) computational complexities

- # of possible paths: 20 DP had to find only: 15

finite state systems

optimal control problem



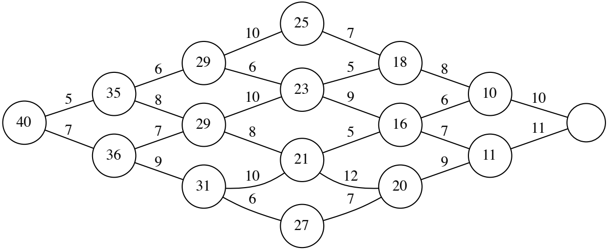
(naive) computational complexities

- # of possible paths: 20 DP had to find only: 15

$n \times n$	4	5	6	7	8
# of paths	20	70	252	724	2632
DP computations	15	24	35	48	63

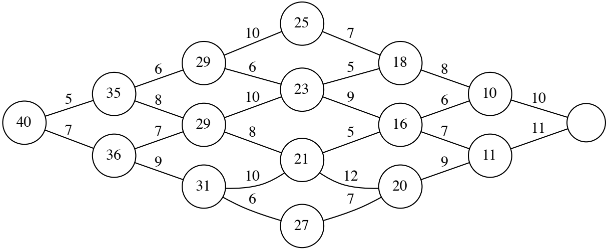
finite state systems

optimal control problem



finite state systems

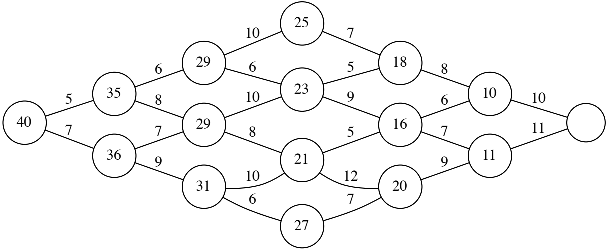
optimal control problem



define $V: X \to \mathbb{R}$:

finite state systems

optimal control problem



define $V : X \to \mathbb{R}$:

$$
\begin{array}{llll}
V(x_1) = 40 & V(x_5) = 29 & V(x_9) = 21 & V(x_{13}) = 20 \\
V(x_2) = 35 & V(x_6) = 31 & V(x_{10}) = 27 & V(x_{14}) = 10 \\
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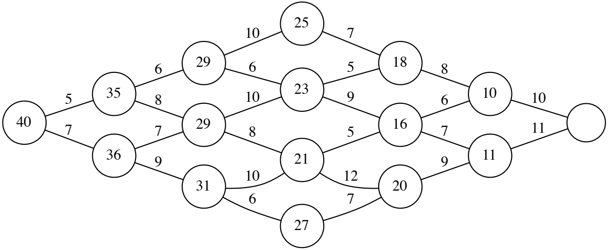
finite state systems

optimal control problem



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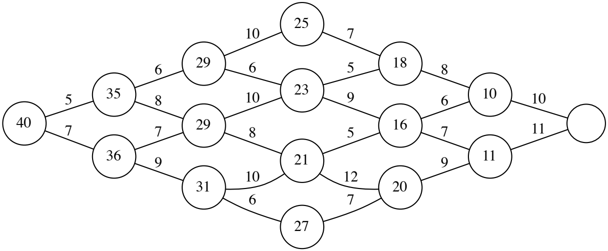
$V(x_i)$ provides the optimal cost starting from x_i

finite state systems

optimal control problem



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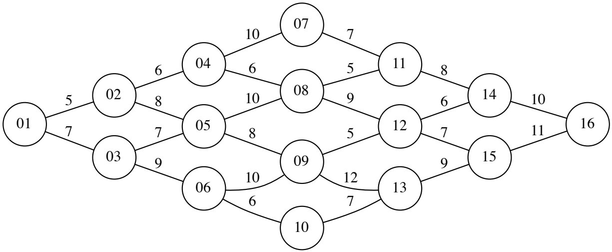
finite state systems

optimal control problem



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$\vdots$	
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