

Advanced Control Systems Engineering I:

Optimal Control

contents

optimal control

- ▶ nonlinear dynamical systems and linear approximations
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- ▶ optimal control of linear systems
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finite state systems

optimal control problem

$$\begin{aligned}x(t) \in X = \{x_1, x_2, \dots, x_n\} \quad & u(t) \in U = \{u_1, u_2, \dots, u_m\} \\ & t \in \{t_0, t_0 + 1, \dots, t_f\}\end{aligned}$$

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_f-1} \ell(x(\tau), u(\tau)) + \ell_f(x(t_f))$$

$$\inf_{u(\cdot)} J(t_0, x_0; u(\cdot))$$

finite state systems

optimal control problem

$$x(t) \in X = \{x_1, x_2, \dots, x_n\} \quad u(t) \in U = \{u_1, u_2, \dots, u_m\}$$
$$t \in \{t_0, t_0 + 1, \dots, t_f\}$$

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_f-1} \ell(x(\tau), u(\tau)) + \ell_f(x(t_f))$$

$$\inf_{u(\cdot)} J(t_0, x_0; u(\cdot))$$

$\phi(x_i, u_j)$	u_1	u_2	$\cdots$	u_m
x_1	x_3	x_{n-2}	$\cdots$	x_1
x_2	x_2	x_8	$\cdots$	x_n
$\vdots$				
x_n	x_5	x_{n-7}	$\cdots$	x_2

finite state systems

optimal control problem

$$x(t) \in X = \{x_1, x_2, \dots, x_n\} \quad u(t) \in U = \{u_1, u_2, \dots, u_m\}$$
$$t \in \{t_0, t_0 + 1, \dots, t_f\}$$

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_f-1} \ell(x(\tau), u(\tau)) + \ell_f(x(t_f))$$

$$\inf_{u(\cdot)} J(t_0, x_0; u(\cdot))$$

$\ell(x_i, u_j)$	u_1	u_2	$\dots$	u_m
x_1	3	2	$\dots$	-1
x_2	2	-2	$\dots$	6
$\vdots$				
x_n	-1	5	$\dots$	1.2

	$\ell_f(x_i)$
x_1	3
x_2	2
$\vdots$	
x_n	-1

finite state systems

optimal control problem

$$x(t) \in X = \{x_1, x_2, \dots, x_n\}$$

$$u(t) \in U = \{u_1, u_2, \dots, u_m\}$$

$$t \in T = \{t_0, t_0 + 1, \dots, t_f\}$$

$$\phi : X \times U \rightarrow X \qquad x(t+1) = \phi(x(t), u(t))$$

$$\ell : X \times U \rightarrow \mathbb{R} \qquad \ell(x(t), u(t))$$

$$\ell_f : X \rightarrow \mathbb{R} \qquad \ell_f(x(t_f))$$

Let $x(t_0) = x_0 \in X$ be given, and consider the optimal control problem:

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_f-1} \ell(x(\tau), u(\tau)) + \ell_f(x(t_f))$$

$$\inf_{\substack{u(\tau) \in U \\ \tau \in T}} J(t_0, x_0; u(\cdot))$$

the cost-to-go

finite state systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_f-1} \ell(x(\tau), u(\tau)) + \ell_f(x(t_f))$$

the cost-to-go

finite state systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_f-1} \ell(x(\tau), u(\tau)) + \ell_f(x(t_f))$$

$$V : T \times X \rightarrow \mathbb{R} \qquad V(t, x) = \inf_{\substack{u(\tau) \in U \\ \tau \in \{t, t+1, \dots, t_f\}}} J(t, x; u(\cdot))$$

the cost-to-go

finite state systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_f-1} \ell(x(\tau), u(\tau)) + \ell_f(x(t_f))$$

$$V : T \times X \rightarrow \mathbb{R} \qquad V(t, x) = \inf_{\substack{u(\tau) \in U \\ \tau \in \{t, t+1, \dots, t_f\}}} J(t, x; u(\cdot))$$

Bellman equation:

$$V(t_f, x) = \ell_f(x) \qquad \text{for all } x \in X$$

$$V(t, x) = \inf_{u \in U} \{ \ell(x, u) + V(t+1, \phi(x, u)) \}$$

$$\text{for all } x \in X \text{ and all } t \in \{t_0, t_0+1, \dots, t_f-1\}$$

state feedback implementation

finite state systems

Let $V : T \times X \rightarrow \mathbb{R}$ be a solution to

$$V(t_f, x) = \ell_f(x) \quad \text{for all } x \in X$$

$$V(t, x) = \inf_{u \in U} \{ \ell(x, u) + V(t+1, \phi(x, u)) \}$$

for all $x \in X$ and all $t \in \{t_0, t_{0+1}, \dots, t_f - 1\}$

state feedback implementation

finite state systems

Let $V : T \times X \rightarrow \mathbb{R}$ be a solution to

$$V(t_f, x) = \ell_f(x) \quad \text{for all } x \in X$$

$$V(t, x) = \inf_{u \in U} \{ \ell(x, u) + V(t+1, \phi(x, u)) \}$$

for all $x \in X$ and all $t \in \{t_0, t_{0+1}, \dots, t_f - 1\}$

For a given x at time t , the optimal input $u(t)$ is given as

$$u(t) = \arg \min_{u \in U} \{ \ell(x, u) + V(t+1, \phi(x, u)) \}$$

state feedback implementation

finite state systems

Let $V : T \times X \rightarrow \mathbb{R}$ be a solution to

$$V(t_f, x) = \ell_f(x) \quad \text{for all } x \in X$$

$$V(t, x) = \inf_{u \in U} \{ \ell(x, u) + V(t+1, \phi(x, u)) \}$$

for all $x \in X$ and all $t \in \{t_0, t_{0+1}, \dots, t_f - 1\}$

For a given x at time t , the optimal input $u(t)$ is given as

$$u(t) = \arg \min_{u \in U} \{ \ell(x, u) + V(t+1, \phi(x, u)) \}$$

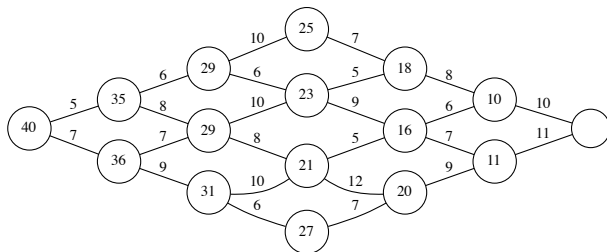
State feedback control:

$$u(t) = u(x(t)) = \arg \min_{u \in U} \underbrace{\{ \ell(x(t), u) + V(t+1, \phi(x(t), u)) \}}_{\text{computed using the measured state } x(t)}$$

$$x(t+1) = \phi(x(t), u(t)) \quad x(t_0) = x_0 \in X$$

example00

finite state systems



define $V : X \rightarrow \mathbb{R}$:

$$V(x_1) = 40 \quad V(x_5) = 29 \quad V(x_9) = 21 \quad V(x_{13}) = 20$$

$$V(x_2) = 35 \quad V(x_6) = 31 \quad V(x_{10}) = 27 \quad V(x_{14}) = 10$$

$$V(x_3) = 36 \quad V(x_7) = 25 \quad V(x_{11}) = 18 \quad V(x_{15}) = 11$$

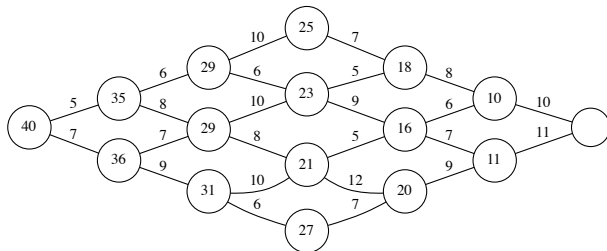
$$V(x_4) = 29 \quad V(x_8) = 23 \quad V(x_{12}) = 16 \quad V(x_{16}) = 0$$

$V(x_i)$ provides the optimal cost starting from x_i

V : cost-to-go

example00

finite state systems



State feedback control:

$$u(t) = u(x(t)) = \arg \min_{u \in U} \underbrace{\{\ell(x(t), u) + V(t+1, \phi(x(t), u))\}}_{\text{computed using the measured state } x(t)}$$

$$x(t+1) = \phi(x(t), u(t)) \quad x(t_0) = x_0 \in X$$

example

finite state systems

$$x(t) \in X = \{x_1, x_2, x_4, x_5\} \quad u(t) \in U = \{u_1, u_2, u_3\} \quad t \in T = \{1, 2, 3\}$$

$$\phi : X \times U \rightarrow X$$

$$\ell : X \times U \rightarrow \mathbb{R}$$

$$\ell_f : X \rightarrow \mathbb{R}$$

$$x(t+1) = \phi(x(t), u(t))$$

$$\ell(x(t), u(t))$$

$$\ell_f(x(t_f))$$

$\phi(x_i, u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i, u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$\ell_f(x_i)$
x_1	1
x_2	2
x_3	3
x_4	4
x_5	5

example

finite state systems

$$x(t) \in X = \{x_1, x_2, x_4, x_5\} \quad u(t) \in U = \{u_1, u_2, u_3\} \quad t \in T = \{1, 2, 3\}$$

Let $x(1) \in X$ be given, and consider the optimal control problem:

$$\begin{aligned} J(1, x(1); u(\cdot)) &= \sum_{\tau=1}^2 \ell(x(\tau), u(\tau)) + \ell_f(x(3)) \\ &= \ell(x(1), u(1)) + \ell(x(2), u(2)) + \ell_f(x(3)) \end{aligned}$$

$$\inf_{u(1), u(2) \in U} J(1, x(1); u(\cdot))$$

$\phi(x_i, u_j)$	u_1	u_2	u_3	$\ell(x_i, u_j)$	u_1	u_2	u_3		$\ell_f(x_i)$
x_1	x_3	x_1	x_5	x_1	1	5	3	x_1	1
x_2	x_4	x_3	x_2	x_2	4	1	2	x_2	2
x_3	x_2	x_5	x_4	x_3	2	3	1	x_3	3
x_4	x_1	x_2	x_1	x_4	3	4	5	x_4	4
x_5	x_5	x_4	x_3	x_5	5	2	4	x_5	5

example

finite state systems

$$x(t) \in X = \{x_1, x_2, x_4, x_5\} \quad u(t) \in U = \{u_1, u_2, u_3\} \quad t \in T = \{1, 2, 3\}$$

Let $x(1) \in X$ be given, and consider the optimal control problem:

$$\begin{aligned} J(1, x(1); u(\cdot)) &= \sum_{\tau=1}^2 \ell(x(\tau), u(\tau)) + \ell_f(x(3)) \\ &= \ell(x(1), u(1)) + \ell(x(2), u(2)) + \ell_f(x(3)) \\ &\inf_{u(1), u(2) \in U} J(1, x(1); u(\cdot)) \end{aligned}$$

$\phi(x_i, u_j)$	u_1	u_2	u_3	$\ell(x_i, u_j)$	u_1	u_2	u_3	$\ell_f(x_i)$	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1	x_3	x_1	x_5	x_1	1	5	3	x_1	1	x_1	
x_2	x_4	x_3	x_2	x_2	4	1	2	x_2	2	x_2	
x_3	x_2	x_5	x_4	x_3	2	3	1	x_3	3	x_3	
x_4	x_1	x_2	x_1	x_4	3	4	5	x_4	4	x_4	
x_5	x_5	x_4	x_3	x_5	5	2	4	x_5	5	x_5	

Bellman equation

example

$$V(t_f, x) = \ell_f(x) \quad \text{for all } x \in X$$

$$V(t, x) = \inf_{u \in U} \{ \ell(x, u) + V(t+1, \phi(x, u)) \}$$

$$\text{for all } x \in X \text{ and all } t \in \{t_0, t_{0+1}, \dots, t_f - 1\}$$

Bellman equation

example

$$V(t_f, x) = \ell_f(x) \quad \text{for all } x \in X$$

$$V(t, x) = \inf_{u \in U} \{ \ell(x, u) + V(t+1, \phi(x, u)) \}$$

$$\text{for all } x \in X \text{ and all } t \in \{t_0, t_{0+1}, \dots, t_f - 1\}$$

$$t = 3 : \quad V(3, x) = \ell_f(x) \quad \text{for all } x \in X$$

$$t = 2 : \quad V(2, x) = \inf_{u \in U} \{ \ell(x, u) + V(3, \phi(x, u)) \} \quad \text{for all } x \in X$$

$$t = 1 : \quad V(1, x) = \inf_{u \in U} \{ \ell(x, u) + V(2, \phi(x, u)) \} \quad \text{for all } x \in X$$

the cost-to-go

example

$$t = 3 : \qquad V(3, x) = \ell_f(x) \qquad \text{for all } x \in X$$

	$\ell_f(x_i)$
x_1	1
x_2	2
x_3	3
x_4	4
x_5	5

	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1			
x_2			
x_3			
x_4			
x_5			

the cost-to-go

example

$$t = 3 : \qquad V(3, x) = \ell_f(x) \qquad \text{for all } x \in X$$

	$\ell_f(x_i)$
x_1	1
x_2	2
x_3	3
x_4	4
x_5	5

	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1			1
x_2			2
x_3			3
x_4			4
x_5			5

the cost-to-go

example

$$t = 2 : \quad V(2, x) = \inf_{u \in U} \{ \ell(x, u) + V(3, \phi(x, u)) \} \quad \text{for all } x \in X$$

$\phi(x_i, u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i, u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1			1
x_2			2
x_3			3
x_4			4
x_5			5

the cost-to-go

example

$$t = 2 : \quad V(2, x) = \inf_{u \in U} \{ \ell(x, u) + V(3, \phi(x, u)) \} \quad \text{for all } x \in X$$

$$V(2, x_1) = \inf_{u \in U} \underbrace{\{ \ell(x_1, u) \}}_{\substack{1 \\ 5 \\ 3}} + \underbrace{V(3, \phi(x_1, u))}_{\substack{3 \\ 1 \\ 5}} = 4$$

$\phi(x_i, u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i, u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1			1
x_2			2
x_3			3
x_4			4
x_5			5

the cost-to-go

example

$$t = 2 : \quad V(2, x) = \inf_{u \in U} \{ \ell(x, u) + V(3, \phi(x, u)) \} \quad \text{for all } x \in X$$

$$V(2, x_2) = \inf_{u \in U} \underbrace{\{ \ell(x_2, u) \}}_{\substack{4 \\ 1 \\ 2}} + \underbrace{V(3, \phi(x_2, u))}_{\substack{4 \\ 3 \\ 2}} = 4$$

$\phi(x_i, u_j)$	u_1	u_2	u_3	$\ell(x_i, u_j)$	u_1	u_2	u_3	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1	x_3	x_1	x_5	x_1	1	5	3	x_1		1
x_2	x_4	x_3	x_2	x_2	4	1	2	x_2		2
x_3	x_2	x_5	x_4	x_3	2	3	1	x_3		3
x_4	x_1	x_2	x_1	x_4	3	4	5	x_4		4
x_5	x_5	x_4	x_3	x_5	5	2	4	x_5		5

the cost-to-go

example

$$t = 2 : \quad V(2, x) = \inf_{u \in U} \{ \ell(x, u) + V(3, \phi(x, u)) \} \quad \text{for all } x \in X$$

$$V(2, x_3) = \inf_{u \in U} \underbrace{\ell(x_3, u)}_{\substack{2 \\ 3 \\ 1}} + \underbrace{V(3, \phi(x_3, u))}_{\substack{2 \\ 5 \\ 4}} = 4$$

$\phi(x_i, u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i, u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1			1
x_2			2
x_3			3
x_4			4
x_5			5

the cost-to-go

example

$$t = 2 : \quad V(2, x) = \inf_{u \in U} \{ \ell(x, u) + V(3, \phi(x, u)) \} \quad \text{for all } x \in X$$

$$V(2, x_4) = \inf_{u \in U} \{ \underbrace{\ell(x_4, u)}_{\substack{3 \\ 4 \\ 5}} + \underbrace{V(3, \phi(x_4, u))}_{\substack{1 \\ 2 \\ 1}} \} = 4$$

$\phi(x_i, u_j)$	u_1	u_2	u_3	$\ell(x_i, u_j)$	u_1	u_2	u_3	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1	x_3	x_1	x_5	x_1	1	5	3	x_1		1
x_2	x_4	x_3	x_2	x_2	4	1	2	x_2		2
x_3	x_2	x_5	x_4	x_3	2	3	1	x_3		3
x_4	x_1	x_2	x_1	x_4	3	4	5	x_4		4
x_5	x_5	x_4	x_3	x_5	5	2	4	x_5		5

the cost-to-go

example

$$t = 2 : \quad V(2, x) = \inf_{u \in U} \{ \underbrace{\ell(x, u)}_5 + \underbrace{V(3, \phi(x, u))}_{\begin{smallmatrix} 5 \\ 4 \\ 3 \end{smallmatrix}} \} \quad \text{for all } x \in X$$

$$V(2, x_5) = \inf_{u \in U} \{ \underbrace{\ell(x_5, u)}_{\begin{smallmatrix} 5 \\ 2 \\ 4 \end{smallmatrix}} + \underbrace{V(3, \phi(x_5, u))}_{\begin{smallmatrix} 5 \\ 4 \\ 3 \end{smallmatrix}} \} = 6$$

$\phi(x_i, u_j)$	u_1	u_2	u_3	$\ell(x_i, u_j)$	u_1	u_2	u_3	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1	x_3	x_1	x_5	x_1	1	5	3	x_1		1
x_2	x_4	x_3	x_2	x_2	4	1	2	x_2		2
x_3	x_2	x_5	x_4	x_3	2	3	1	x_3		3
x_4	x_1	x_2	x_1	x_4	3	4	5	x_4		4
x_5	x_5	x_4	x_3	x_5	5	2	4	x_5		5

the cost-to-go

example

$$t = 2 : \quad V(2, x) = \inf_{u \in U} \{ \underbrace{\ell(x, u)}_{\substack{5 \\ 2 \\ 4}} + \underbrace{V(3, \phi(x, u))}_{\substack{5 \\ 4 \\ 3}} \} \quad \text{for all } x \in X$$

$$V(2, x_5) = \inf_{u \in U} \{ \underbrace{\ell(x_5, u)}_{\substack{5 \\ 2 \\ 4}} + \underbrace{V(3, \phi(x_5, u))}_{\substack{5 \\ 4 \\ 3}} \} = 6$$

$\phi(x_i, u_j)$	u_1	u_2	u_3	$\ell(x_i, u_j)$	u_1	u_2	u_3	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1	x_3	x_1	x_5	x_1	1	5	3	x_1	4	1
x_2	x_4	x_3	x_2	x_2	4	1	2	x_2	4	2
x_3	x_2	x_5	x_4	x_3	2	3	1	x_3	4	3
x_4	x_1	x_2	x_1	x_4	3	4	5	x_4	4	4
x_5	x_5	x_4	x_3	x_5	5	2	4	x_5	6	5

the cost-to-go

example

$$t = 1 : \quad V(1, x) = \inf_{u \in U} \{ \ell(x, u) + V(2, \phi(x, u)) \} \quad \text{for all } x \in X$$

$\phi(x_i, u_j)$	u_1	u_2	u_3	$\ell(x_i, u_j)$	u_1	u_2	u_3		$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1	x_3	x_1	x_5	x_1	1	5	3	x_1		4	1
x_2	x_4	x_3	x_2	x_2	4	1	2	x_2		4	2
x_3	x_2	x_5	x_4	x_3	2	3	1	x_3		4	3
x_4	x_1	x_2	x_1	x_4	3	4	5	x_4		4	4
x_5	x_5	x_4	x_3	x_5	5	2	4	x_5		6	5

the cost-to-go

example

$$t = 1 : \quad V(1, x) = \inf_{u \in U} \{ \ell(x, u) + V(2, \phi(x, u)) \} \quad \text{for all } x \in X$$

$$V(1, x_1) = \inf_{u \in U} \underbrace{\{ \ell(x_1, u) \}}_{\substack{1 \\ 5 \\ 3}} + \underbrace{V(2, \phi(x_1, u))}_{\substack{4 \\ 4 \\ 6}} = 5$$

$\phi(x_i, u_j)$	u_1	u_2	u_3	$\ell(x_i, u_j)$	u_1	u_2	u_3	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1	x_3	x_1	x_5	x_1	1	5	3	x_1	4	1
x_2	x_4	x_3	x_2	x_2	4	1	2	x_2	4	2
x_3	x_2	x_5	x_4	x_3	2	3	1	x_3	4	3
x_4	x_1	x_2	x_1	x_4	3	4	5	x_4	4	4
x_5	x_5	x_4	x_3	x_5	5	2	4	x_5	6	5

the cost-to-go

example

$$t = 1 : \quad V(1, x) = \inf_{u \in U} \{ \ell(x, u) + V(2, \phi(x, u)) \} \quad \text{for all } x \in X$$

$$V(1, x_1) = \inf_{u \in U} \underbrace{\{ \ell(x_1, u) \}}_{\substack{1 \\ 5 \\ 3}} + \underbrace{V(2, \phi(x_1, u))}_{\substack{4 \\ 4 \\ 6}} = 5$$

$\phi(x_i, u_j)$	u_1	u_2	u_3	$\ell(x_i, u_j)$	u_1	u_2	u_3		$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1	x_3	x_1	x_5	x_1	1	5	3	x_1	5	4	1
x_2	x_4	x_3	x_2	x_2	4	1	2	x_2	5	4	2
x_3	x_2	x_5	x_4	x_3	2	3	1	x_3	5	4	3
x_4	x_1	x_2	x_1	x_4	3	4	5	x_4	7	4	4
x_5	x_5	x_4	x_3	x_5	5	2	4	x_5	6	6	5

sample response

example

consider the initial condition $x(1) = x_5$:

$\phi(x_i, u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i, u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$\ell_{\text{f}}(x_i)$
x_1	1
x_2	2
x_3	3
x_4	4
x_5	5

	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1	5	4	1
x_2	5	4	2
x_3	5	4	3
x_4	7	4	4
x_5	6	6	5

sample response

example

consider the initial condition $x(1) = x_5$:

$$x(1) = x_5$$

$\phi(x_i, u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i, u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$\ell_{\text{f}}(x_i)$
x_1	1
x_2	2
x_3	3
x_4	4
x_5	5

	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1	5	4	1
x_2	5	4	2
x_3	5	4	3
x_4	7	4	4
x_5	6	6	5

sample response

example

consider the initial condition $x(1) = x_5$:

$$x(1) = x_5$$

$$u(1) = \arg \min_{u \in U} \left\{ \underbrace{\ell(x_5, u)}_{\substack{5 \\ 2 \\ 4}} + \underbrace{V(2, \phi(x_5, u))}_{\substack{6 \\ 4 \\ 4}} \right\} = u_2$$

$\phi(x_i, u_j)$	u_1	u_2	u_3
x_1	x_3	x_1	x_5
x_2	x_4	x_3	x_2
x_3	x_2	x_5	x_4
x_4	x_1	x_2	x_1
x_5	x_5	x_4	x_3

$\ell(x_i, u_j)$	u_1	u_2	u_3
x_1	1	5	3
x_2	4	1	2
x_3	2	3	1
x_4	3	4	5
x_5	5	2	4

	$\ell_{\text{f}}(x_i)$
x_1	1
x_2	2
x_3	3
x_4	4
x_5	5

	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$
x_1	5	4	1
x_2	5	4	2
x_3	5	4	3
x_4	7	4	4
x_5	6	6	5

sample response

example

consider the initial condition $x(1) = x_5$:

$$x(1) = x_5$$

$$u(1) = \arg \min_{u \in U} \left\{ \underbrace{\ell(x_5, u)}_{\substack{5 \\ 2 \\ 4}} + \underbrace{V(2, \phi(x_5, u))}_{\substack{6 \\ 4 \\ 4}} \right\} = u_2$$

$$x(2) = \phi(x_5, u_2) = x_4$$

$\phi(x_i, u_j)$	u_1	u_2	u_3	$\ell(x_i, u_j)$	u_1	u_2	u_3	$\ell_f(x_i)$	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$	
x_1	x_3	x_1	x_5	x_1	1	5	3	x_1	1	5	4	1
x_2	x_4	x_3	x_2	x_2	4	1	2	x_2	2	5	4	2
x_3	x_2	x_5	x_4	x_3	2	3	1	x_3	3	5	4	3
x_4	x_1	x_2	x_1	x_4	3	4	5	x_4	4	7	4	4
x_5	x_5	x_4	x_3	x_5	5	2	4	x_5	5	6	6	5

sample response

example

consider the initial condition $x(1) = x_5$:

$$x(1) = x_5$$

$$u(1) = \arg \min_{u \in U} \left\{ \underbrace{\ell(x_5, u)}_{\substack{5 \\ 2 \\ 4}} + \underbrace{V(2, \phi(x_5, u))}_{\substack{6 \\ 4 \\ 4}} \right\} = u_2$$

$$x(2) = \phi(x_5, u_2) = x_4$$

$$u(2) = \arg \min_{u \in U} \left\{ \underbrace{\ell(x_4, u)}_{\substack{3 \\ 4 \\ 5}} + \underbrace{V(3, \phi(x_4, u))}_{\substack{1 \\ 2 \\ 1}} \right\} = u_1$$

$\phi(x_i, u_j)$	u_1	u_2	u_3	$\ell(x_i, u_j)$	u_1	u_2	u_3	$\ell_f(x_i)$	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$		
x_1	x_3	x_1	x_5	x_1	1	5	3	x_1	1	x_1	5	4	1
x_2	x_4	x_3	x_2	x_2	4	1	2	x_2	2	x_2	5	4	2
x_3	x_2	x_5	x_4	x_3	2	3	1	x_3	3	x_3	5	4	3
x_4	x_1	x_2	x_1	x_4	3	4	5	x_4	4	x_4	7	4	4
x_5	x_5	x_4	x_3	x_5	5	2	4	x_5	5	x_5	6	6	5

sample response

example

consider the initial condition $x(1) = x_5$:

$$x(1) = x_5$$

$$u(1) = \arg \min_{u \in U} \left\{ \underbrace{\ell(x_5, u)}_{\substack{5 \\ 2 \\ 4}} + \underbrace{V(2, \phi(x_5, u))}_{\substack{6 \\ 4 \\ 4}} \right\} = u_2$$

$$x(2) = \phi(x_5, u_2) = x_4$$

$$u(2) = \arg \min_{u \in U} \left\{ \underbrace{\ell(x_4, u)}_{\substack{3 \\ 4 \\ 5}} + \underbrace{V(3, \phi(x_4, u))}_{\substack{1 \\ 2 \\ 1}} \right\} = u_1$$

$$x(3) = \phi(x_4, u_1) = x_1$$

$\phi(x_i, u_j)$	u_1	u_2	u_3	$\ell(x_i, u_j)$	u_1	u_2	u_3	$\ell_f(x_i)$	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$	
x_1	x_3	x_1	x_5	x_1	1	5	3	x_1	1	5	4	1
x_2	x_4	x_3	x_2	x_2	4	1	2	x_2	2	5	4	2
x_3	x_2	x_5	x_4	x_3	2	3	1	x_3	3	5	4	3
x_4	x_1	x_2	x_1	x_4	3	4	5	x_4	4	7	4	4
x_5	x_5	x_4	x_3	x_5	5	2	4	x_5	5	6	6	5

sample response

example

consider the initial condition $x(1) = x_5$:

$$\begin{aligned}\min_{u(1), u(2) \in U} J(1, x_5; u(\cdot)) &= \ell(x_5, u_2) + \ell(x_4, u_1) + \ell_f(x_1) \\ &= 2 + 3 + 1 = 6 = V(1, x_5)\end{aligned}$$

$\phi(x_i, u_j)$	u_1	u_2	u_3	$\ell(x_i, u_j)$	u_1	u_2	u_3	$\ell_f(x_i)$	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$	
x_1	x_3	x_1	x_5	x_1	1	5	3	x_1	1	5	4	1
x_2	x_4	x_3	x_2	x_2	4	1	2	x_2	2	5	4	2
x_3	x_2	x_5	x_4	x_3	2	3	1	x_3	3	5	4	3
x_4	x_1	x_2	x_1	x_4	3	4	5	x_4	4	7	4	4
x_5	x_5	x_4	x_3	x_5	5	2	4	x_5	5	6	6	5

computer exercise

finite state systems

computer exercise

finite state systems

Bellman equation:

$$V(t_f, x) = \ell_f(x) \quad \text{for all } x \in X$$

$$V(t, x) = \inf_{u \in U} \{ \ell(x, u) + V(t+1, \phi(x, u)) \}$$

for all $x \in X$ and all $t \in \{t_0, t_{0+1}, \dots, t_f - 1\}$

$\phi(x_i, u_j)$	u_1	u_2	u_3	$\ell(x_i, u_j)$	u_1	u_2	u_3	$\ell_f(x_i)$	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$	
x_1	x_3	x_1	x_5	x_1	1	5	3	x_1	1	5	4	1
x_2	x_4	x_3	x_2	x_2	4	1	2	x_2	2	5	4	2
x_3	x_2	x_5	x_4	x_3	2	3	1	x_3	3	5	4	3
x_4	x_1	x_2	x_1	x_4	3	4	5	x_4	4	7	4	4
x_5	x_5	x_4	x_3	x_5	5	2	4	x_5	5	6	6	5

computer exercise

finite state systems

Bellman equation:

$$V(t_f, x) = \ell_f(x) \quad \text{for all } x \in X$$

$$V(t, x) = \inf_{u \in U} \{ \ell(x, u) + V(t+1, \phi(x, u)) \}$$

for all $x \in X$ and all $t \in \{t_0, t_{0+1}, \dots, t_f - 1\}$

$\phi(x_i, u_j)$	u_1	u_2	u_3	$\ell(x_i, u_j)$	u_1	u_2	u_3	$\ell_f(x_i)$	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$	
x_1	x_3	x_1	x_5	x_1	1	5	3	x_1	1	5	4	1
x_2	x_4	x_3	x_2	x_2	4	1	2	x_2	2	5	4	2
x_3	x_2	x_5	x_4	x_3	2	3	1	x_3	3	5	4	3
x_4	x_1	x_2	x_1	x_4	3	4	5	x_4	4	7	4	4
x_5	x_5	x_4	x_3	x_5	5	2	4	x_5	5	6	6	5

data: ϕ , $n \times m$ table (matrix)

ℓ , $n \times m$ table (matrix)

ℓ_f , $n \times 1$ table (vector)

computer exercise

finite state systems

Bellman equation:

$$V(t_f, x) = \ell_f(x) \quad \text{for all } x \in X$$

$$V(t, x) = \inf_{u \in U} \{ \ell(x, u) + V(t+1, \phi(x, u)) \}$$

for all $x \in X$ and all $t \in \{t_0, t_{0+1}, \dots, t_f - 1\}$

$\phi(x_i, u_j)$	u_1	u_2	u_3	$\ell(x_i, u_j)$	u_1	u_2	u_3	$\ell_f(x_i)$	$V(1, x_i)$	$V(2, x_i)$	$V(3, x_i)$	
x_1	x_3	x_1	x_5	x_1	1	5	3	x_1	1	5	4	1
x_2	x_4	x_3	x_2	x_2	4	1	2	x_2	2	5	4	2
x_3	x_2	x_5	x_4	x_3	2	3	1	x_3	3	5	4	3
x_4	x_1	x_2	x_1	x_4	3	4	5	x_4	4	7	4	4
x_5	x_5	x_4	x_3	x_5	5	2	4	x_5	5	6	6	5

data: ϕ , $n \times m$ table (matrix)

ℓ , $n \times m$ table (matrix)

ℓ_f , $n \times 1$ table (vector)

output: V , $n \times m$ table (matrix)

computational complexity

finite state systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_f-1} \ell(x(\tau), u(\tau)) + \ell_f(x(t_f))$$

brute force search:

$$O(|U|^{|T|} \times |T| \times |X|)$$

DP algorithm:

$$O(|U| \times |X| \times |T|)$$

$$x(t) \in X = \{x_1, x_2, \dots, x_n\}$$

$|X|$: cardinality of X

$$u(t) \in U = \{u_1, u_2, \dots, u_m\}$$

$|U|$

$$t \in T = \{t_0, t_0 + 1, \dots, t_f\}$$

$|T|$

contents

optimal control

- ▶ nonlinear dynamical systems and linear approximations
- ▶ dynamic programming
- ▶ the principle of optimality
- ▶ optimal control of finite state systems
- ▶ optimal control of discrete-time systems
- ▶ optimal control of continuous-time systems
- ▶ optimal control of linear systems
- ▶ decentralized optimal control
 - ▶ decentralization and integration via mechanism design