Advanced Control Systems Engineering I:

Optimal Control

contents

optimal control systems

- nonlinear dynamical systems and linear approximations
- dynamic programming
- the principle of optimality
- optimal control of finite state systems
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- optimal control of linear systems
- decentralized optimal control
  - decentralization and integration via mechanism design

finite state systems

optimal control problem

$$\begin{aligned} x(t) &\in X = \{x_1, x_2, \ldots, x_n\} \\ u(t) &\in U = \{u_1, u_2, \ldots, u_m\} \\ t &\in T = \{t_0, t_0 + 1, \ldots, t_\text{f}\} \end{aligned}$$

finite state systems

optimal control problem

$$x(t) \in X = \{x_1, x_2, \ldots, x_n\}$$
$$u(t) \in U = \{u_1, u_2, \ldots, u_m\}$$
$$t \in T = \{t_0, t_0 + 1, \ldots, t_f\}$$

$$\phi : \; X \times U \to X \qquad x(t+1) = \phi(x(t), u(t))$$
$$\ell : \; X \times U \to \mathbb{R} \qquad \ell(x(t), u(t))$$
$$\ell_f : \; X \to \mathbb{R} \qquad \ell_f(x(t_f))$$

finite state systems

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$$\phi : X \times U \to X \qquad x(t+1) = \phi(x(t), u(t))$$
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$$\ell_\mathrm{f} : X \to \mathbb{R} \qquad \ell_\mathrm{f}(x(t_\mathrm{f}))$$

Let $x(t_0) = x_0 \in X$ be given, and consider the optimal control problem:

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

$$\inf_{\substack{u(\tau)\in U \\ \tau\in T}} J(t_0, x_0; u(\cdot))$$

state feedback implementation

finite state systems

Let $V: T \times X \to \mathbb{R}$ be a solution to

$$V(t_\mathrm{f}, x) = \ell_\mathrm{f}(x) \qquad \text{for all } x \in X$$

$$V(t, x) = \inf_{u \in U} \{\ell(x, u) + V(t+1, \phi(x, u))\}$$

$$\text{for all } x \in X \text{ and all } t \in \{t_0, t_{0+1}, \dots, t_\mathrm{f} - 1\}$$

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For a given x at time t, the optimal input $u(t)$ is given as

$$u(t) = \arg\min_{u \in U} \{\ell(x, u) + V(t+1, \phi(x, u))\}$$

state feedback implementation

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State feedback control:

$$u(t) = u(x(t)) = \arg\min_{u \in U} \{\underbrace{\ell(x(t), u) + V(t+1, \phi(x(t), u))}_{\text{computed using the measured state } x(t)}\}$$

$$x(t+1) = \phi(x(t), u(t)) \qquad x(t_0) = x_0 \in X$$

finite state systems

optimal control problem

$$
\begin{aligned}
x(t) &\in X = \{x_1, x_2, \ldots, x_n\} \\
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\end{aligned}
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\begin{aligned}
\phi :&\ X \times U \to X \qquad & x(t+1) = \phi(x(t), u(t)) \\
\ell :&\ X \times U \to \mathbb{R} \qquad & \ell(x(t), u(t)) \\
\ell_\mathrm{f} :&\ X \to \mathbb{R} \qquad & \ell_\mathrm{f}(x(t_\mathrm{f}))
\end{aligned}
$$

Let $x(t_0) = x_0 \in X$ be given, and consider the optimal control problem:

$$
J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))
$$

$$
\inf_{\substack{u(\tau)\in U \\ \tau\in T}} J(t_0, x_0; u(\cdot))
$$

discrete-time systems

optimal control problem

$$x(t+1) = f(x(t), u(t)) \qquad x(t_0) = x_0 \qquad t \in T = \{t_0, t_0+1, \ldots, t_f\}$$

$$x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

discrete-time systems

optimal control problem

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For a given $x(t_0) = x_0 \in \mathbb{R}^n$

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_f))$$

discrete-time systems

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$$\inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in T}} J(t_0, x_0; u(\cdot))$$

the cost-to-go

discrete-time systems

Let $x(t_0) = x_0 \in X$ be given, and consider the optimal control problem:

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

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define the cost-to-go:

the cost-to-go

discrete-time systems

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$$\inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in T}} J(t_0, x_0; u(\cdot))$$

define the cost-to-go:

$$V : \; T \times \mathbb{R}^n \to \mathbb{R} \qquad\qquad V(t,x) = \inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in\{t,t+1,\ldots,t_\mathrm{f}\}}} J(t, x; u(\cdot))$$

the cost-to-go

discrete-time systems

Let $x(t_0) = x_0 \in X$ be given, and consider the optimal control problem:

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

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computing the cost-to-go $V(t_0, x_0)$ from the initial state x_0 at the initial time t_0 essentially amounts to minimize the cost $J(t_0, x_0; u(\cdot))$

the cost-to-go

discrete-time systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

$$V : \; T \times \mathbb{R}^n \to \mathbb{R} \qquad V(t,x) = \inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in\{t,t+1,\ldots,t_\mathrm{f}\}}} J(t,x;u(\cdot))$$

If $t = t_\mathrm{f}$:

the cost-to-go

discrete-time systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_f-1} \ell(x(\tau), u(\tau)) + \ell_f(x(t_f))$$

$$V : \ T \times \mathbb{R}^n \to \mathbb{R} \qquad V(t,x) = \inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in\{t,t+1,\dots,t_f\}}} J(t,x;u(\cdot))$$

If $t = t_f$:

$$\begin{aligned} V(t_f, x) &= \inf_{u(t_f)\in\mathbb{R}^m} J(t_f, x; u(t_f)) \\ &= \inf_{u(t_f)\in\mathbb{R}^m} \underbrace{\ell_f(x)}_{\text{independent of } u(t_f)} = \ell_f(x) \end{aligned}$$

the cost-to-go

discrete-time systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

$$V : \; T \times \mathbb{R}^n \to \mathbb{R} \qquad\qquad V(t, x) = \inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in\{t,t+1,\dots,t_\mathrm{f}\}}} J(t, x; u(\cdot))$$

If $t = t_\mathrm{f} - 1$:

the cost-to-go

discrete-time systems

If $t = t_\mathrm{f} - 1$:

$$
\begin{aligned}
V(t_\mathrm{f}-1, x) &= \inf_{u(t_\mathrm{f}-1), u(t_\mathrm{f}) \in \mathbb{R}^m} J(t_\mathrm{f}-1, x; u(\cdot)) \\
&= \inf_{u(t_\mathrm{f}-1), u(t_\mathrm{f}) \in \mathbb{R}^m} \{ \underbrace{\ell(x(t_\mathrm{f}-1), u(t_\mathrm{f}-1))}_{\text{independent of } u(t_\mathrm{f})} + \underbrace{\ell_\mathrm{f}(x(t_\mathrm{f}))}_{\substack{\text{depend on both} \\ u(t_\mathrm{f}-1) \text{ and } u(t_\mathrm{f})}} \} \\
&= \inf_{u(t_\mathrm{f}-1) \in \mathbb{R}^m} \{ \ell(x, u(t_\mathrm{f}-1)) + \underbrace{\inf_{u(t_\mathrm{f}) \in \mathbb{R}^m} \ell_\mathrm{f}(x(t_\mathrm{f}))}_{= V(t_\mathrm{f}, x(t_\mathrm{f})) = V(t_\mathrm{f}, f(x, u(t_\mathrm{f}-1)))} \} \\
&= \inf_{u(t_\mathrm{f}-1) \in \mathbb{R}^m} \{ \ell(x, u(t_\mathrm{f}-1)) + V(t_\mathrm{f}, f(x, u(t_f - 1))) \} \\
&= \inf_{u \in \mathbb{R}^m} \{ \ell(x, u) + V(t_\mathrm{f}, f(x, u)) \}
\end{aligned}
$$

the cost-to-go

discrete-time systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_f-1} \ell(x(\tau), u(\tau)) + \ell_f(x(t_f))$$

$$V : \; T \times \mathbb{R}^n \to \mathbb{R} \qquad V(t, x) = \inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in\{t, t+1, \ldots, t_f\}}} J(t, x; u(\cdot))$$

For $t < t_f$:

the cost-to-go

discrete-time systems

For $t < t_f$:

$$V(t,x) = \inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in\{t,t+1,\ldots,t_f\}}} J(t,x;u(\cdot))$$

$$= \inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in\{t,t+1,\ldots,t_f\}}} \left\{\sum_{\tau=t}^{t_f-1} \ell(x(\tau),u(\tau)) + \ell_f(x(t_f))\right\}$$

$$= \inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in\{t,t+1,\ldots,t_f\}}} \Big\{ \underbrace{\ell(x,u(t))}_{\substack{\text{independent of } u(\tau), \\ \tau\in\{t+1,t+2,\ldots,t_f\}}} + \underbrace{\sum_{\tau=t+1}^{t_f-1} \ell(x(\tau),u(\tau)) + \ell_f(x(t_f))}_{\text{depend on all } u(\tau),\ \tau\in\{t,t+1,\ldots,t_f}$$

$$= \inf_{u(t)\in\mathbb{R}^m}\Big\{\ell(x,u(t)) + \underbrace{\inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in\{t+1,t+2,\ldots,t_f\}}} \Big\{\sum_{\tau=t+1}^{t_f-1} \ell(x(\tau),u(\tau)) + \ell_f(x(t_f)}_{=V(t+1,x(t+1))=V(t+1,f(x,u(t)))}$$

$$= \inf \{\ell(x,u(t)) + V(t+1,f(x,u(t)))\}$$

the cost-to-go

discrete-time systems

$$
\begin{aligned}
V(t,x) &= \inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in\{t,t+1,\dots,t_\mathrm{f}\}}} J(t,x;u(\cdot)) \\
&= \inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in\{t,t+1,\dots,t_\mathrm{f}\}}} \Big\{\sum_{\tau=t}^{t_\mathrm{f}-1} \ell(x(\tau),u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))\Big\} \\
&= \inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in\{t,t+1,\dots,t_\mathrm{f}\}}} \Big\{ \underbrace{\ell(x,u(t))}_{\substack{\text{independent of } u(\tau), \\ \tau\in\{t+1,t+2,\dots,t_\mathrm{f}\}}} + \underbrace{\sum_{\tau=t+1}^{t_\mathrm{f}-1} \ell(x(\tau),u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))}_{\text{depend on all } u(\tau),\ \tau\in\{t,t+1,\dots,t_\mathrm{f}\}} \Big\} \\
&= \inf_{u(t)\in\mathbb{R}^m} \Big\{\ell(x,u(t)) + \underbrace{\inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in\{t+1,t+2,\dots,t_\mathrm{f}\}}} \Big\{\sum_{\tau=t+1}^{t_\mathrm{f}-1} \ell(x(\tau),u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))\Big\}}_{=\,V(t+1,x(t+1))\,=\,V(t+1,f(x,u(t)))}\Big\} \\
&= \inf_{u(t)\in\mathbb{R}^m} \{\ell(x,u(t)) + V(t+1,f(x,u(t)))\} \\
&= \inf_{u\in\mathbb{R}^m} \{\ell(x,u) + V(t+1,f(x,u))\}
\end{aligned}
$$

Bellman equation:

discrete-time systems

$$J(t_0, x_0; u(\cdot)) = \sum_{\tau=t_0}^{t_\mathrm{f}-1} \ell(x(\tau), u(\tau)) + \ell_\mathrm{f}(x(t_\mathrm{f}))$$

$$V : \; T \times \mathbb{R}^n \to \mathbb{R} \qquad V(t,x) = \inf_{\substack{u(\tau)\in\mathbb{R}^m \\ \tau\in\{t,t+1,\ldots,t_\mathrm{f}\}}} J(t,x;u(\cdot))$$

Bellman equation:

$$V(t_\mathrm{f}, x) = \ell_\mathrm{f}(x) \qquad \text{for all } x \in \mathbb{R}^n$$

$$V(t,x) = \inf_{u\in\mathbb{R}^m} \{\ell(x,u) + V(t+1, f(x,u))\}$$

for all $x \in \mathbb{R}^n$ and all $t \in \{t_0, t_{0+1}, \ldots, t_\mathrm{f} - 1\}$

state feedback implementation

discrete-time systems

Let $V: \; T \times \mathbb{R}^n \to \mathbb{R}$ be a solution to

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state feedback implementation

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Let $V : T \times \mathbb{R}^n \to \mathbb{R}$ be a solution to

$$V(t_f, x) = \ell_f(x) \qquad \text{for all } x \in \mathbb{R}^n$$

$$V(t, x) = \inf_{u \in \mathbb{R}^m} \{\ell(x, u) + V(t+1, f(x, u))\}$$

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For a given x at time t, the optimal input $u(t)$ is given as

$$u(t) = \arg\min_{u \in \mathbb{R}^m} \{\ell(x, u) + V(t+1, f(x, u))\}$$

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$$u(t) = u(x(t)) = \arg\min_{u \in \mathbb{R}^m} \{\underbrace{\ell(x(t), u) + V(t+1, f(x(t), u))}_{\text{computed using the measured state } x(t)}\}$$

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